Feb 4, 2021. Lecture 9

We are continuing with our proof of the following thm:

If X is a complete variety, T a reduced connected scheme, $\mathcal{L}$ a line bundle on $X_T := X \times_k T$, such that $L_t = \mathcal{L}|_{X \times \{t\}}$ is trivial for every t, then $\exists!$ (up to isomorphism) line bundle M on T such that $\mathcal{L} = p_2^* M$.

Recall $k = \bar{k}$. By a scheme we mean a finite type scheme over k. A point on T will mean a closed point, and since $k = \bar{k}$, and T is of finite type, this is the same as a k-rational point of T.

Recall, we picked an affine open cover $\mathcal{U} = \{U_\alpha\}$ on X, and set $\mathcal{V} = \mathcal{U} \times_k T = \{V_\alpha\} = \{U_\alpha \times_k T\}$. We also had reduced to the case $T = \operatorname{Spec} A$, so that $\{V_\alpha\}$ is also an affine open cover. $C^\cdot := C^\cdot(\mathcal{V}, \mathcal{L})$. We had a commutative diagram with each row complex:

$$\begin{tikzcd}
0 \arrow[r] & P^0 \arrow[r] \arrow[d, "\varphi^0"] & P^1 \arrow[r] \arrow[d, "\varphi^1"] & P^2 \arrow[r] \arrow[d, "\varphi^2"] & \cdots \arrow[r] & P^n \arrow[r] \arrow[d, "\varphi^n"] & 0 \\
0 \arrow[r] & C^0 \arrow[r] & C^1 \arrow[r] & C^2 \arrow[r] & \cdots \arrow[r] & C^n \arrow[r] & 0
\end{tikzcd}$$

such that

- $\varphi : P^\bullet \longrightarrow C^\bullet$ is a quism
- $P^\bullet$ is a complex of f.g projective A-modules.

We noted that $C^\cdot$ is a flat complex and hence, for every $M \in \mathrm{Mod}_A$,

$$\varphi \otimes_A M : P^\cdot \otimes_A M \longrightarrow C^\cdot \otimes_A M$$

is a quism. Moreover, since $H^i(C^\cdot \otimes_A M) = H^i(X_T, \mathcal{L} \otimes_A M)$ $\forall M$, therefore $H^i(P^\cdot \otimes_A M) \xrightarrow{\sim} H^i(X_T, \mathcal{L} \otimes_A M)$.

Let

$$Q = \operatorname{coker}\left((P^1)^\vee \xrightarrow{{}^t\delta^0} (P^0)^\vee\right).$$

Then we argued that

$$H^0(X_T, \mathcal{L} \otimes_A M) \xrightarrow{\sim} \operatorname{Hom}_A(Q, M)$$
$$\forall\, M \in \operatorname{Mod}_A.$$

This is so because the exact sequence

$$(P^1)^\vee \xrightarrow{{}^t\delta^0} (P^0)^\vee \longrightarrow Q \longrightarrow 0$$

gives, on applying the left exact contravariant functor $\operatorname{Hom}_A(-, M)$ to the above exact sequence, the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(Q, M) \longrightarrow P^0 \otimes_A M \xrightarrow{\delta^0 \otimes M} P^1 \otimes_A M.$$

$$\left(\text{since } \operatorname{Hom}_A(P^\vee, M) = P \otimes_A M \text{ for } P \text{ projective}\right)$$

Hence

$$\operatorname{Hom}_A(Q, M) = H^0(P^\cdot \otimes M) = H^0(X_T, \mathcal{L} \otimes_A M).$$

Let $t \in T$. Setting $M = k(t)$ in the above, we get

$$\begin{aligned}
\operatorname{Hom}_A(Q, k(t)) &= H^0(X_T, \mathcal{L} \otimes_A k(t)) \\
&= H^0(X_T, \mathcal{L}|_{X \times \{t\}}) \\
&= H^0(X_T, L_t) \\
&= H^0(X, L_t) \\
&\simeq H^0(X, \mathcal{O}_X)
\end{aligned}$$

since $L_t \simeq \mathcal{O}_X$ by our hypothesis.

This means

$$\dim_k \mathrm{Hom}_A(Q, k(t)) = \dim_k H^0(X, \mathcal{O}_X) = 1.$$

Now $k(t) = A/m_t$, m_t = max'l ideal of $t \in T$.

Hence $\mathrm{Hom}_A(Q, k(t)) = \mathrm{Hom}_A(Q/m_tQ, k(t))$

$$= \mathrm{Hom}_k(Q/m_tQ, k)$$

It follows that

$$\dim_{k(t)} Q/m_tQ = 1 \quad \forall\, t \in T.$$

i.e.

$$\dim_{k(t)} Q \otimes_A k(t) = 1.$$

Since T is reduced and connected, this means $\tilde{Q}$ is a line bundle on T.

Shrinking T, can assume $Q \simeq A$ as an A-module and all P^i are free A-modules. In particular

$$H^0(X_T, \mathcal{L}) = \mathrm{Hom}_A(Q, A)$$

is free of rank one.

This was a local computation on the base scheme T. Globalise to get $\mathcal{N} = p_{2*}\mathcal{L}$ is a line bundle on T.

Now (p_2^*, p_{2*}) is an adjoint pair, i.e., for $\mathcal{F} \in T_{qc}$ and $\mathcal{G} \in (X_T)_{qc}$ we have

$$\mathrm{Hom}_{X_T}(p_2^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_T(\mathcal{F}, (p_2)_*\mathcal{G}).$$

In particular, we have a canonical map

$$p_2^{*}\mathcal{N} = p_2^{*}\, p_{2*}\mathcal{L} \longrightarrow \mathcal{L}$$

as the adjoint to the identity map $p_{2*}\mathcal{L} \longrightarrow p_{2*}\mathcal{L}$.

$$\left(\operatorname{Hom}(p_2^{*}\, p_{2*}\mathcal{L}, \mathcal{L}) = \operatorname{Hom}(p_{2*}\mathcal{L}, p_{2*}\mathcal{L}) \right)$$

In practical terms the map is (if $T = \operatorname{Spec} A$) the usual map $H^0(X_T, \mathcal{L}) \otimes_A \mathcal{O}_{X_T} \longrightarrow \mathcal{L}$.

If we show that the map just defined

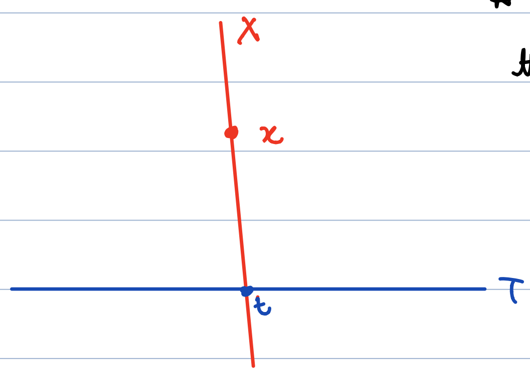
$$p_2^{*}\mathcal{N} \longrightarrow \mathcal{L} \quad (\text{i.e. } p_2^{*}\, p_{2*}\mathcal{L} \longrightarrow \mathcal{L})$$

is surjective, we are done, for a surjective map of line bundles is an isomorphism.

By Nakayama's lemma, it is enough to show that for $x \in X_T$, the map

$$(p_2^{*}\mathcal{N}) \otimes_{\mathcal{O}_{X_T}} k(x) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_{X_T}} k(x)$$

is surjective. Let $t = p_2(x)$.



A little thought shows that the above is

$$(p_2^{*}\mathcal{N})\big|_{X \times \{t\}} \otimes_{\mathcal{O}_X} k(x)$$

$$\longrightarrow L_t \otimes_{\mathcal{O}_X} k(x) = \mathcal{O}_X \otimes_{\mathcal{O}_X} k(x)$$

~~This amounts to showing that~~

$$\cancel{H^0(X_T, \mathcal{L}) \otimes_A k(x) \longrightarrow k(x)}$$

~~is surjective.~~

See more convincing argument below.

Consider the map

$$(*) \quad p_2^* \mathcal{N}\big|_{X\times_k\{t\}} \longrightarrow \mathcal{L}\big|_{X\times_k\{t\}} = L_t.$$

By shrinking T in a neighbourhood of t, we may assume (as before) that T is affine, say $T = \operatorname{Spec} A$, and that we have a rank 1 free A-module Q such that $\operatorname{Hom}_A(Q, M) = H^0(X_T, \mathcal{L}\otimes_A M)$, $M \in \operatorname{Mod}_A$. The map $p_2^*\mathcal{N} = p_2^* p_{2*}\mathcal{L} \longrightarrow \mathcal{L}$ is the natural map

$$H^0(X_T, \mathcal{L}) \otimes_A \mathcal{O}_{X_T} \longrightarrow \mathcal{L}$$

and $(*)$ is the map

$$H^0(X_T, \mathcal{L}) \otimes_A \mathcal{O}_{X\times\{t\}} \longrightarrow L_t$$

This fits into a diagram

$$(+) \quad \begin{tikzcd} H^0(X_T, \mathcal{L}) \otimes_A \mathcal{O}_{X_T} \arrow[d] \arrow[dr] & \\ H^0(X_T, \mathcal{L}) \otimes_A \mathcal{O}_{X\times\{t\}} \arrow[r, "(*)"'] & L_t \end{tikzcd}$$

The global sections of the south-east pointing arrow gives CD:

$$\begin{tikzcd} H^0(X_T, \mathcal{L}) \arrow[r] & H^0(X_T, L_t) \\ \operatorname{Hom}_A(Q, A) \arrow[u, "\wr"] \arrow[r] & \operatorname{Hom}_A(Q, k(t)) \arrow[u, "\wr"] \end{tikzcd}$$

since $\operatorname{Hom}_A(Q, -) \longrightarrow H^0(X_T, \mathcal{L}\otimes_A -)$ is a natural transformation.

Since Q is a projective A-module, the horizontal arrow at the bottom is surjective. Hence

$$H^0(X_T, \mathcal{L}) \twoheadrightarrow H^0(X, L_t) \quad (**)$$

is surjective (note $H^0(X_T, L_t) = H^0(X, L_t)$).

Since L_t is trivial (by hypothesis), it has a nowhere vanishing section σ. Since (**) is surjective there is a section $\tilde{\sigma}$ of $\mathcal{L}$ such that $\tilde{\sigma}|_{X\times\{t\}} = L_t$. Consider the commutative diagram (†). Since σ is nowhere vanishing, one every affine open subscheme U of X_T, it is nowhere vaishing on $U\cap(X\times\{t\})$ and generates the line bundle L_t on $U\cap(X\times\{t\})$. Since $\tilde{\sigma}|_U$ maps to $\sigma|_{U\cap(X\times\{t\})}$, it follows that the south-east pointing arrow in (†) is surjective. It follows that (*) is surjective. An immediate consequence is that $H^0(X_T,\mathcal{L})\otimes_A \mathcal{O}_{X_T}\otimes_{\mathcal{O}_{X_T}} k(x) \longrightarrow \mathcal{L}\otimes_{\mathcal{O}_{X_T}} k(x)$ is surjective, i.e. $p_2^*\mathcal{N}\otimes_{\mathcal{O}_{X_T}} k(x) \longrightarrow \mathcal{L}\otimes_{\mathcal{O}_{X_T}} k(x)$ is surjective. Since this is true for all $x\in X_T$, by Nakayama's lemma, $p_2^*\mathcal{N}\longrightarrow\mathcal{L}$ is surjective, and since this is a surjection of line bundles, it must be an isomorphism.

Uniqueness is seen as follows. Let $\mathcal{N}'$ be another line bundle on T such that $p_2^*\mathcal{N}'\simeq\mathcal{L}$. Pick a point $x\in X$. Then

$$\mathcal{N}' = p_2^*\mathcal{N}'|_{\{x\}\times T} \xrightarrow{\sim} \mathcal{L}|_{\{x\}\times T} \xrightarrow{\sim} p_2^*\mathcal{N}|_{\{x\}\times T} = \mathcal{N}.$$

This gives uniquess.

Remarks: For these remarks k is no longer alg. closed.

1. If X is a geometrically connected complete variety. Then a line bundle L on X is trivial if and only if its pull back to some X_K is trivial where

K is a field extension of k.

$L_K := q^*L$

$$\begin{tikzcd} X_K \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\square"] & X \quad L \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

By flat base change

$$H^0(X_K, L_K) = H^0(X, L) \otimes_k K.$$

Hence $\dim_k H^0(X,L) = \dim_K H^0(X_K, L_K) = 1.$ ← Since L_K is trivial.

From this it is easy to see L is trivial.