Feb 2, 2021.

Lecture 8

Recall that if S is an integral (noetherian) scheme and $\mathcal{F}$ a coherent $\mathcal{O}_S$-module such that

$$s \longmapsto \dim_{k(s)} \mathcal{F} \otimes_{\mathcal{O}_S} k(s) \qquad s \in S$$

is constant, then $\mathcal{F}$ is locally free of rank equal to the common dimension above.

The proof is an exercise in Nakayama's lemma, and here is how it goes. Without loss of generality, we may assume $S = \operatorname{Spec} A$. Let $\mathcal{F} = \widetilde{M}$, $M \in \mathrm{Mod}_A$. M is a finitely generated A-module. Let ~~$\mathfrak{p} \in A$~~ $\mathfrak{p} \subseteq A$ be a prime ideal (translation: let $s \in S$ be a point), and $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ the residue field at $\mathfrak{p}$. Let n be the common dimension of $M \otimes_A k(\mathfrak{q})$, as $\mathfrak{q}$ varies over $\operatorname{Spec} A$. Pick a basis $e_1, \dots, e_n$ of $M \otimes_A k(\mathfrak{p})$, and lift $e_1, \dots, e_n$ to generators $\frac{m_1}{s_1}, \dots, \frac{m_n}{s_n} \in M_\mathfrak{p}$ (Recall, $M_\mathfrak{p} \longrightarrow M \otimes_A k(\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$ is surjective, since $A_\mathfrak{p} \longrightarrow k(\mathfrak{p})$ is surjective and $\otimes$-product is right exact).
By clearing denominators we may assume, the generators are of the form $\frac{m_1}{1}, \dots, \frac{m_n}{1}$, with $m_i \in M$.
Now consider $m_1, \dots, m_n \in M$.

We have an A-module map

$$A^n \longrightarrow M$$

which sends the i^{th} member of the standard basis of A^n to m_i. This is a surjection at $\mathfrak{p}$, i.e. $A^n_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}$ is surjective (by Nakayama's lemma). In fact there exists $t \in A - \mathfrak{p}$ such that

$$t^{-1}A^n \longrightarrow t^{-1}M$$

(Let $C = \operatorname{coker}(A^n \to M)$. Then $C_{\mathfrak{p}} = 0$. Since C f.g., $\exists\, t \in A-\mathfrak{p}$ s.t. $t^{-1}C = 0$)

is surjective. Replace A by $t^{-1}A$ if necessary and assume

$$A^n \longrightarrow M$$

is surjective. Let $K = \ker(A^n \to M)$. If $\mathfrak{q} \in \operatorname{Spec} A$, we have the exact sequence

$$K \otimes_A k(\mathfrak{q}) \longrightarrow A^n \otimes_A k(\mathfrak{q}) \longrightarrow M \otimes_A k(\mathfrak{q}) \longrightarrow 0$$

Since $\dim M \otimes_A k(\mathfrak{q}) = n$, it follows that $A^n \otimes_A k(\mathfrak{q}) \longrightarrow M \otimes_A k(\mathfrak{q})$ is an isomorphism.

So the image of $K \otimes_A k(\mathfrak{q})$ in $A^n \otimes_A k(\mathfrak{q})$ is zero. It follows that $K \subseteq \mathfrak{q}(A^n)$ $\forall\, \mathfrak{q} \in \operatorname{Spec} A$. Since A is integral, $K = 0$. //

P.T.O $\longrightarrow$ D

Proposition: Suppose T is an integral k-scheme and X a complete k-variety. Let $\mathcal{L}$ be a line bundle on $X \times_k T$ and $L_t = \mathcal{L}|_{X \times_k \{t\}}$. Suppose L_t is trivial for all $t \in T$. Then there exists a line bundle $\mathcal{M}$ on T such that $\mathcal{L} = p_2^* \mathcal{M}$, where p_1, p_2 are the projections from $X \times_k T$ to X and T respectively.

<u>Remark (long)</u>: Let us reduce to the case $T = \operatorname{Spec} A$. Suppose the statement is true whenever T is affine.

Let $U = \operatorname{Spec} A$ be an affine open subset of T. We know that $\mathcal{L}|_{p_2^{-1}(U)} = p_2^* M_U$, where M_U is a line bundle on U, and we write p_2 for all projections to the 2nd factor. By uniqueness $M_U|_{U \cap V} \simeq M_V|_{U \cap V}$ for U, V affine open subschemes of T. Moreover, one checks that this identification is compatible on the $\cap$'s of three affine open subschemes. (Use a k-rational point $x_0 \in X$, and restrict $\mathcal{L}|_{p_2^{-1}(U \cap V)}$ to $\{x_0\} \times U \cap V$ etc, to see the patching) Hence we get $\mathcal{M}$ on T. (Really, the patching is occurring on the copy of T given by $\{x_0\} \times_k T$, and one is using the fact that $\mathcal{L}|_{\{x_0\} \times_k U} \cong M_U$).

From now on $T = \operatorname{Spec} A$.

Let $\mathcal{U} = \{U_\alpha\}$ be a finite affine open cover of X. Let $\mathcal{V} = \{V_\alpha\}$ be the affine open cover $\mathcal{V} \times T = \{U_\alpha \times_k T\}$.

on $X \times_k T$.

Let $C^\bullet = C^\bullet(\mathcal{V}, \mathcal{L})$ be the Čech complex of $\mathcal{L}$ with respect to $\mathcal{V}$. Since $\mathcal{U}$ is a finite, so is $\mathcal{V}$, and hence $C^\bullet$ is bounded. Moreover, since X is complete, therefore $H^j(C^\bullet)$ is f.g. as an A-module for $j \geq 0$. So from general homological algebra, we can find a bounded complex of f.g. projective modules $P^\bullet$, $P^i = 0$, $i < 0$, and a q-isomorphism $P^\bullet \xrightarrow{\varphi} C^\bullet$, with $P^1, P^2, \ldots$ free:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots & \longrightarrow & P^n & \longrightarrow & 0 \\
 & & \varphi^0 \big\downarrow & & \big\downarrow \varphi^1 & & & & \big\downarrow \varphi^n & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & \cdots & \longrightarrow & C^n & \longrightarrow & 0
\end{array}$$

By shrinking T around a given point, if necessary, we may assume $P^\bullet$ is a complex of f.g. free modules.

From general considerations we discussed last lecture, we know

$$H^i(C^\bullet \otimes_A M) \xrightarrow{\sim} H^i(X \times_k T, \mathcal{L} \otimes_A M)$$

where

$$\mathcal{L} \otimes_A M := \mathcal{L} \otimes_{\mathcal{O}_{X \times_k T}} p_2^* \widetilde{M}.$$

Since $\mathcal{L}$ is a line bundle on $X \times_k T$, therefore $\Gamma(V, \mathcal{L})$ is flat as $\Gamma(V, \mathcal{O}_{X \times_k T})$-module for every affine open subscheme V of $X \times_k T$. Moreover since $X \longrightarrow \operatorname{Spec} k$ is a flat map (k is a field!) therefore

$X \times_k T \longrightarrow T$ is flat (being a base change). In particular $V \longrightarrow T$ is flat. Hence $\Gamma(V, \mathcal{L})$ is a flat A-module.

All this goes to show that $C^\bullet$ is a flat complex of A-modules.

Since $\phi : P^\bullet \longrightarrow C^\bullet$ is a q-iso of flat complexes of flat modules, therefore for any $M \in \mathrm{Mod}_A$,

$$\phi \otimes M : P^\bullet \otimes_A M \longrightarrow C^\bullet \otimes_A M$$

is also a quism (= q-iso).

In particular

$$H^i(P^\bullet \otimes_A M) \xrightarrow{\sim} H^i(X_T, \mathcal{L} \otimes_A M)$$

where

$$X_T := X \times_k T$$

Now let $Q = \mathrm{coker}\left(P_1^\vee \xrightarrow{(\delta^0)^t} P_0^\vee\right)$

Have

$$P_1^\vee \xrightarrow{(\delta^0)^t} P_0^\vee \longrightarrow Q \longrightarrow 0 \quad \text{(exact)}$$

Then for $M \in \mathrm{Mod}_A$, have

$$0 \to \mathrm{Hom}_A(Q, M) \longrightarrow \mathrm{Hom}_A(P_0^\vee, M) \longrightarrow \mathrm{Hom}_A(P_1^\vee, M) \quad \text{(exact)}$$

Now for any f.g. free (or f.g. projective) module P, we have

$$\mathrm{Hom}_A(P^\vee, M) = \mathrm{Hom}_A(A, P \otimes_A M) = P \otimes_A M.$$

Hence we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(Q, M) \longrightarrow P^0 \otimes_A M \longrightarrow P^1 \otimes_A M$$

In particular, since $H^i(P^\bullet \otimes_A M) = H^i(X_T, \mathcal{L} \otimes_A M)$, we see that

$$\mathrm{Hom}_A(Q, M) = \Gamma(X_T, \mathcal{L} \otimes_A M) = H^0(X_T, \mathcal{L} \otimes_A M)$$

We will use this and the result we proved at the beginning of the lecture to show that $\mathrm{Hom}_A(Q, A)$ is locally free of rank 1 (i.e. a line bundle) and this is what we will need to prove the theorem.