Jan 28, 2001

Lecture 7

Working over $k = \bar{k}$
A = abelian variety $/k$.

Last time we showed the following.

If L is a line bundle on A with $H^0(A, L) \neq 0$ then L is ample if and only if $K(L)$ is finite.

Let $U = \operatorname{Spec} A$ be a non-empty affine open subscheme of A with $0 \in U$. Let D be any effective divisor such that $U \cup D = A$. Such a D clearly exists. By translating D and U, we may assume $e \notin D$. (e = identity for A).

Let $L = \mathcal{O}(D)$. Since D is effective $H^0(A, L) \neq 0$. Let s be a section of L whose zero locus is D. Note that $s(e) \neq 0$ since $e \notin D$

It follows that $s|_{K(L)^0}$ is non-vanishing, since $e \in K(L)^0$, and $s(e) \neq 0$. Now $L|_{K(L)^0}$ is trivial (we saw this in an earlier lecture). This means $s|_{K(L)^0}$ is nowhere vanishing, because $K(L)^0$ is a complete variety and hence sections $\mathcal{O}_{K(L)^0}$ are either zero or non-zero elements from k.

This means $K(L)^0 \cap D = \phi$. Hence $K(L)^0 \subseteq U = \operatorname{Spec} A$. A complete connected subvariety of an affine scheme$/_k$ is necessarily a point. This means $K(L)$ is finite, whence L is ample. Hence A is projective. //

Suppose $T \hookrightarrow \operatorname{Spec} A$ where T is a complete variety over k (and A is a f.g. k-algebra), with $\hookrightarrow$ being a closed immersion. Then T is affine, since all closed subschemes of an affine scheme are affine (they are all of the form $\operatorname{Spec}(A/I)$ where I is an ideal of A). Hence $T = \operatorname{Spec}(\Gamma(T, \mathcal{O}_T))$. Now $T \longrightarrow \operatorname{Spec} k$ is a proper map, and hence $\Gamma(T, \mathcal{O}_T)$ is a f.g. k-module, i.e, it is a finite k-module, i.e. a finite dim'l k-vector space. Moreover $\Gamma(T, \mathcal{O}_T)$ is an integral domain and hence is a field. But $k = \bar{k}$, whence $\Gamma(T, \mathcal{O}_T) = k$. Hence $T = \operatorname{Spec}(\Gamma(T, \mathcal{O}_T)) = \operatorname{Spec} k$ is a point.

Lemma: Suppose X is a complete k-variety and L a line bundle on X. Then L is trivial if and only if $H^0(A, L) \neq 0$ and $H^0(A, L^{-1}) \neq 0$.

Proof: If L is trivial, then this is obvious.
Conversely suppose $H^0(L)$ and $H^0(L^{-1})$ are non-zero.
Let $0 \neq \sigma \in H^0(L)$ and $0 \neq \tau \in H^0(L^{-1})$.
Then σ can be identified with a map

$$\mathcal{O}_X \xrightarrow{\sigma} L$$

and τ with

$$L \xrightarrow{\tau} \mathcal{O}_X.$$

Let U be the open locus on which neither σ nor τ is zero. Consider the composite

$$\mathcal{O}_X \xrightarrow{\sigma} L \xrightarrow{\tau} \mathcal{O}_X .$$

This composite does not vanish on U, and hence is a non-zero element of $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = k$. Hence it is nowhere vanishing and $\tau \circ \sigma$ is an isomorphism. This means τ is surjective. Now L and $\mathcal{O}_X$ are line bundles, whence τ is an isomorphism. //

Proposition: Suppose $\mathcal{L}$ is a line bundle on $X \times_k T$ with X complete, and T a k-scheme. Let $L_t := \mathcal{L}|_{X \times_k \{t\}}$ for $t \in T$. Then the locus on T where L_t is free is a closed subset of T.

Proof:

By semi-continuity, the locus where $\dim_k H^0(X, L_t) > 0$ is closed, as is the locus where $\dim_k H^0(X, L_t^{-1}) > 0$. //

Proposition: Suppose T is an integral k-scheme and X a complete k-variety. Let $\mathcal{L}$ be a line bundle on $X \times_k T$ and $L_t = \mathcal{L}|_{X \times_k \{t\}}$. Suppose L_t is trivial for all $t \in T$. Then there exists a line bundle $\mathcal{M}$ on T such that $\mathcal{L} = p_2^* \mathcal{M}$, where p_1, p_2 are the projections from $X \times_k T$ to X and T respectively.

Remark: The line bundle $\mathcal{M}$ on T is unique (up to isomorphism). This can be seen by picking a k-rational point $x_0 \in X$, and noting that $\mathcal{L}|_{\{x_0\} \times_k T} = p_2^* \mathcal{M}|_{\{x_0\} \times_k T} = \mathcal{M}$.

Proof:

Without loss of generality we may assume T is affine, say $T = \operatorname{Spec} A$. Pick an affine open cover $\mathcal{U} = \{U_i\}$ of $X \times_k T$, and consider $C^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{L})$. We have a natural isomorphism $\check{H}^i(\mathcal{U}, \mathcal{L}) \xrightarrow{\sim} H^i(X \times_k T, \mathcal{L})$, $i \geq 0$.
Moreover, since $X \times_k T \longrightarrow T$ is proper, $H^i(X \times_k T, \mathcal{L})$ is a finitely generated A-module for all i. We can assume $\mathcal{U}$ is a finite cover, and hence $H^i(X \times_k T, \mathcal{L}) = 0$ for $i > n_0$ for some n_0.

Then basic homological algebra says that there is complex of free modules, $L^\bullet$, $L^m = 0$ $m \geq n_0$, and a quasi-isomorphism

$$\varphi : L^\bullet \longrightarrow C^\bullet$$

Moreover, since $C^\bullet$ is flat over A, $\varphi \otimes_A M : L^\bullet \otimes_A M \to C^\bullet \otimes_A M$ is a quasi-isomorphism for any $M \in \mathrm{Mod}_A$.

Note that $H^i(C^\bullet \otimes_A M) = H^i(X \times_k T, \mathcal{L} \otimes_A M)$

Here $\mathcal{L} \otimes_A M$ is short hand for $\mathcal{L} \otimes p^* \tilde{M}$.

Note $$C^p(\mathcal{U}, \mathcal{F} \otimes_A M) = \bigoplus_{0 \leq i_0 < \dots < i_p} (\mathcal{F} \otimes_A M)(U_{i_0 \dots i_p})$$

$$= \bigoplus_{0 \leq i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p}) \otimes_A M.$$

$$= C^p(\mathcal{U}, \mathcal{F}) \otimes_A M.$$

From HW 3, we can "truncate" $L^\bullet$ to get a quasi-isomorphism

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots & \longrightarrow & K^{n_0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & & & \downarrow{\scriptstyle \phi^{n_0}} & & \\
0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & & \longrightarrow & C^{n_0} & \longrightarrow & 0
\end{array}$$

with K^0 a projective A-module. Here $K^i = L^i$ for $i \geq 1$, and $K^0 = (\qquad)$.

Since $K^\bullet$ is flat we have

$$H^i(\phi) : H^i(K^\bullet \otimes_A M) \xrightarrow{\sim} H^i(C^\bullet \otimes_A M) = H^i(X \times_k T, R \otimes_A M)$$

$$\forall\, M \in \mathrm{Mod}_A$$