Jan 26, 2021

Lecture 6

We are working over alg. closed field k. A is an abelian var.
Last time we proved the following:

Theorem: Let $f: A \longrightarrow Y$ be a morphism of varieties with Y complete. For $a \in A$ let F_a be the connected component of $f^{-1}f(a)$ containing a. Then with $F = F_0$ we have $F_a = a + F$ $\forall a \in A$. Moreover F is a closed subgroup of A (i.e. F is a sub abelian variety of A).

Remark: F is often called the kernel of the morphism f.

Now, let D be an effective divisor on A and $L = \mathcal{O}(D)$. We know that $2D$ is base point free, i.e. L^2 is generated by global sections. (Idea: $2D \equiv t_a^* D + t_{-a}^* D$ for every $a \in A$, and given $x \in A$, we can find $a \in A$ such that $x \notin t_a^* D \cup t_{-a}^* D$). Let $V = H^0(A, L^2)^*$. We have a natural map

$$A \xrightarrow{f} \mathbb{P}(V)$$

It follows that if $D^* \in |2D|$, there exists a ! hyperplane H^* in $\mathbb{P}(V)$ such that $f^{-1}(H^*) = D^*$.

Have surjection $H^0(L^2) \otimes_k \mathcal{O}_A \twoheadrightarrow L^2$
i.e. $V^* \otimes_k \mathcal{O}_A \longrightarrow L^2$.
Hence map $A \xrightarrow{f} \mathbb{P}(V)$
and $f^* \mathcal{O}(1) \xrightarrow{\sim} L^2$
and $H^0(L^2) = H^0(\mathcal{O}(1))$

In particular, we have a hyperplane H in $\mathbb{P}(V)$ such that $2D = f^{-1}(H)$.

Set theoretically, $2D$ is the same as D. More precisely $(2D)_{red} = D_{red}$.

This means D (thought of in a naive way — as a set or as a reduced scheme) is the disjoint union of fibres. In view of the theorem, set theoretically

$$D = \bigcup_{a \in D} (a + F)$$

where F is the kernel of $f : A \longrightarrow \mathbb{P}(V)$.

It follows that if $x \in F$ then $t_x^* D = D$ (set theoretically).

Let

$$H(D) = \{ x \in A \mid t_x^* D = D \}$$

Equality of sets, not linear equivalence.

Then we have shown that

$$F \subset H(D)$$

Note that if $a \in A$ is such that $D - a \neq D$, then there exists an open neighbourhood $\mathcal{U}$ of a such that $D - u \neq D$ $\forall$ $u \in \mathcal{U}$. It follows that $H(D)$ is closed. In fact since F is connected

$$F \subset H(D)^{\circ} \qquad (*)$$

The connected component of $H(D)$ containing the identity.

Theorem (M.V. Nori): $F = H(D)^0 = K(L)^0$. (Recall, we are assuming $H^0(L) \neq 0$.)

Proof: We have already seen that $F \subset H(D)^0$. Clearly $H(D)^0 \subset K(L)^0$ (this is obvious, since $H(D)$ is clearly contained in $K(L)$). It remains to show that $K(L)^0 \subset F$.

Now $K(L) \subset K(L^2)$. Hence $K(L)^0 \subset K(L^2)^0$. We know that $L^2 |_{K(L^2)^0}$ is trivial. It follows that $L^2 |_{K(L)^0}$ is trivial. Hence $f(K(L)^0)$ is a point (see notes on line bundles and divisors). Hence $K(L)^0 \subset f^{-1}f(0)$. Since $0 \in K(L)^0$, and since $K(L)^0$ is connected, it follows that $K(L)^0 \subset F$. //

Suppose L as above (i.e. $H^0(L) \neq 0$) is ample. Then for some $n \geq 1$, L^{2n} is very ample. It follows that if $f: A \longrightarrow \mathbb{P}(H^0(L^{2n})^*)$ is the resulting map, it is an embedding, whence, from the theorem, $K(L^n)^0$ $\cancel{K(L^{2n})^0}$ is a point. Now $K(L)^0 \subset K(L^n)^0$. Hence $K(L)^0 = \{0\}$. This means $K(L)$ is finite.

$$L \text{ ample}, \ H^0(L) \neq 0 \implies K(L) \text{ is finite.}$$

Conversely suppose $K(L)$ is finite. Let $f: A \to \mathbb{P}(V)$ be as before, with $V = H^0(A, L^2)^*$, and let F be be the kernel of f. Since $K(L)$ is finite, $K(L)^0 = \{0\}$, whence $F = \{0\}$. Now F is the connected component of $f^{-1}f(0)$ containing 0. Also $f^{-1}f(0)$ is the disjoint union of $x+F$, $x \in f^{-1}f(0)$. It follows that $f^{-1}f(0)$ is finite. Moreover, from the earlier theorem of Nori, this means $f^{-1}f(x)$ is finite for every $x \in A$.

Hence the map

$$f : A \longrightarrow \mathbb{P}(V)$$

is a finite map. (<u>Reason</u>: f is a proper map, since $A \xrightarrow{f} \mathbb{P}(V) \longrightarrow \operatorname{Spec} k$ is proper. A proper quasi-finite map is finite (in Hartshorne this is proved by using Stein factorisation)). This means L^2 is ample (see argument below) and hence L is ample.

To see L^2 is ample (when $K(L)$ is finite) pick a coherent $\mathcal{O}_A$-module $\mathcal{F}$. We have to show that $H^i(A, \mathcal{F} \otimes (L^2)^n) = 0$ for $i \geq 1$, $n \gg 0$. This is the so-called cohomological criterion for ampleness (see [H, p.229, Prop. 5.3])

Now

$$H^i(A, \mathcal{F} \otimes (L^2)^n) = H^i(A, \mathcal{F} \otimes f^*\mathcal{O}(n))$$

Since f is finite, it is an affine map. Hence

$$H^i(A, \mathcal{F} \otimes f^*\mathcal{O}(n)) = H^i(\mathbb{P}(V), f_*(\mathcal{F} \otimes f^*\mathcal{O}(n)))$$

$$= H^i(\mathbb{P}(V), f_*\mathcal{F} \otimes \mathcal{O}(n))$$

Since $f_*(\mathcal{F} \otimes f^*\mathcal{O}(n)) = f_*\mathcal{F} \otimes \mathcal{O}(n)$ by proj'n formula.

<u>Added later</u>:
Since f is affine $f_* : A_{qc} \longrightarrow \mathbb{P}(V)_{qc}$ is exact. Let $\mathcal{G} \in A_{qc}$ and $\mathcal{G} \longrightarrow \mathcal{J}^\bullet$ an inj. resln in A_{qc}. Then $f_*\mathcal{G} \longrightarrow f_*\mathcal{J}^\bullet$ is exact, and $f_*\mathcal{J}^\bullet$ is a flasque complex. Hence $H^i(\mathbb{P}(V), f_*\mathcal{G}) = H^i(\Gamma(\mathbb{P}(V), f_*\mathcal{J}^\bullet)) = H^i(\Gamma(A, \mathcal{J}^\bullet)) = H^i(A, \mathcal{G})$.

(If $A \longrightarrow B$ is a map of ring, $M \in Mod_B$, $N \in Mod_A$, then $M \otimes_B (B \otimes_A N) = M \otimes_A N$.)

Now $\mathcal{O}(n)$ is very ample on $\mathbb{P}(V)$, and hence $H^i(f_* \mathcal{F} \otimes \mathcal{O}(n)) = 0$ $n \gg 0$, $i \geq 1$. //

Hence we have proved the following theorem.

Theorem: Let L be a line bundle on A such that $H^0(A, L) \neq 0$. Then L is ample if and only if $K(L)$ is finite.