Jan 21, 2021

Lecture 5

$k = \bar{k}$, A abelian variety.

Last time we were showing the following:

Suppose L is a line bundle with $H^0(A, L) \neq 0$, then $L|_{K(L)}$ is trivial. It is enough to show that $L|_{K(L)^0}$ is trivial. Let $Y = K(L)^0$. Then Y is an abelian variety, in fact an abelian subvariety of A. Moreover $L|_Y$ is homogeneous.

Easy to see that

$$K(t_a^* L) = K(L) \qquad a \in A.$$

Suppose $b \in K(L)$.

$$
\begin{aligned}
& t_b^*(t_a^* L) \otimes t_a^* L^{-1} \\
&= t_a^*(t_b^* L) \otimes L^{-1} \otimes L \otimes t_a^* L^{-1} \\
&= t_a^*(t_b^* L \otimes L^{-1}) \otimes L^{-1} \otimes L \\
&= t_a^*(\mathcal{O}_A) \otimes \mathcal{O}_A = \mathcal{O}_A .
\end{aligned}
$$

So $b \in K(t_a^* L)$. By symmetry we have the result.

Let s be a non-zero section of L. Let D be the zero scheme of s. Then D is an effective Cartier divisor and $L \cong \mathcal{O}(D)$. Let

$$U = \{ a \in A \mid D - a \not\supset Y \}.$$

If $Y \neq A$, U is a non-empty open set. If $Y = A$ then there is nothing to prove, for then $K(L) = A$, i.e., L is homogeneous.

Assume $Y \neq A$. Pick $a \in U$. Then $t_a^*(s) \in H^0(t_a^* L)$ is non-zero, and there is some point of D on which it does not vanish. Hence $t_a^* s|_Y$ is a non-zero section of $t_a^* L|_Y$. However, $t_a^* L|_Y$ is homogeneous on Y, since $K(t_a^* L) = K(L)$. From our earlier results, this means $t_a^* L|_Y \simeq \mathcal{O}_A$.

Now consider the family

$$\mathcal{L} = (p_1 + p_2)^* L|_{Y \times A}.$$

Note that $\mathcal{L}|_{Y \times \{a\}} = t_a^* L|_Y$

Therefore the family $\mathcal{L}$ is trivial on a non-empty open set of the parameter space A. Since the locus of points $a \in A$ on which $\mathcal{L}|_{Y \times \{a\}}$ is trivial is closed, therefore $U = A$, and $t_a^* L|_Y$ is trivial for all $a \in A$. Taking $a = 0$, we get the result. //

<u>Question</u>: Let L (line bdle) be s.t. $H^0(L) \neq 0$. Know that L^2 is generated by global sections (D effective $\Rightarrow 2D$ is base point free). Let

$$A \xrightarrow{f} \mathbb{P}(H^0(L^2)^*) = \mathbb{P}$$

be the resulting map. What can you say about the locus $f(K^0(L))$ in $\mathbb{P}$?

Theorem: Let A be an abelian variety over an alg. closed field k and $f : A \longrightarrow Y$ be a map of k-varieties. For $x \in A$ let F_x be the connected component of $f^{-1}f(x)$ containing x. Then $\exists$ a closed subgroup F of A such that $F_x = x + F$.

Remark: The theorem is due to M.V. Nori.

Proof: Fix $x \in A$. Let

$$\phi : A \times F_x \longrightarrow Y$$

be the map

$$\phi(a, u) = f(a+u).$$

Then $\phi(0, u) = f(x) \quad \forall\, u \in F_x$.

By rigidity we see that

$$\phi(z, u) = f(z+x). \quad \forall\, u \in F_x.$$

In fact

$$f(z - x + F_x) = f(z). \quad \text{———} \quad (*)$$

This is seen as follows:

Know $\phi(z - x, u)$ is constant for $u \in F_x$.

Since $x \in F_x$, this constant is $\phi(z-x, x) = f(z)$.

This proves $(*)$.

Therefore $z - x + F_x = F_z$

In fact, easy to see $z - x + F_x \subset F_z$

Reverse the roles of z and x. Conclusion:

$$z - x + F_x = F_z.$$

In particular, setting $x = 0$, and $F = F_0$, we

get $$F_z = z + F.$$
It remains to show that F is a subgroup of A. Suppose $y \in F$. Then
$$F_{-y} = -y + F$$
Therefore $0 \in F_{-y}$.
Hence $f(0) = f(-y)$ and 0 and $-y$ lie in the same connected component of $f^{-1}f(0)$. It follows that
$$-y + F = F.$$
Hence $z - y \in F \quad \forall\, z, y \in F.$ //

Now suppose L is a line bundle on A such that $H^0(A, L) \neq 0$, and let D be an effective divisor such that $L \simeq \mathcal{O}(D)$. We know such a D exists. We also know that the linear system $|2D|$ is base point free. Hence we have map
$$A \xrightarrow{f} \mathbb{P}(V) \quad \text{where } V = H^0(A, L^2)^*.$$

Recall

Recall, since L^2 is gen'd by global sections, we have a surjection
$$H^0(L^2) \otimes_k \mathcal{O}_A \twoheadrightarrow L^2.$$
Dualize:
$$L^{-2} \hookrightarrow H^0(L^2)^* \otimes_k \mathcal{O}_A.$$
At each point $a \in A$, get a line $\ell = L^{-2}\big|_{\{a\}}$ inside $H^0(L^2)^*$, i.e. a point in $\mathbb{P}(V)$

From the theorem we have a connected subgroup F = connected component of $f^{-1}f(D)$ containing 0, such that $F_x = x + F$, using the notations of the theorem.

Let

$$H(D) = \{ x \in A \mid t_x^* D = D \}.$$

Equality of divisors, not equivalence classes!

This is a closed subgroup of A.

Clearly $H(D) \subseteq K(L)$.

Also $F \subseteq H(D)$.

This is what we will prove next time:

- $F = H(D)^0 = K(L)^0$
- $K(L)$ is finite $\Longleftrightarrow$ L is ample.