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Lecture 4

Let k be an algebraically closed field.

Theorem: Let A be an abelian variety over k. Then A is smooth over k.

Proof: Since $k = \bar{k}$ and A is a variety, A has a non-singular (closed) point $a \in A$. Let $x \in A$.

Consider the map

$$t_{x-a} : A \longrightarrow A$$
$$y \longmapsto x - a + y.$$

This is an isomorphism of varieties, and hence $\mathcal{O}_{A,a} \xrightarrow{\sim} \mathcal{O}_{A,x}$. It follows that $\mathcal{O}_{A,x}$ is a regular local ring. //

Fix an abelian variety A for the rest of this lecture.

Recall

$$\mathrm{Pic}(A) = \text{iso. classes of line bdles on } A.$$

$L \in \mathrm{Pic}(A)$ is said to be homogeneous if $t_x^* L \simeq L$ for all $x \in A$, where $t_x : A \longrightarrow A$ is the translation map $a \longmapsto x + a$.

$$J(A) = \text{iso. classes of homog. line bdles on } A.$$

Denoted $\mathrm{Pic}^0(A)$ in most books on abelian varieties.

Summary of where we are:

1. Let $\mathcal{L}$ be a family of line bundles on A parametrised by a k-scheme T, i.e. $\mathcal{L}$ is a line bundle on $X \times_k T$. Write L_t for $\mathcal{L}|_{X \times \{t\}}$, t a k-rational point on T. $\{L_t\}_{t \in T}$ is an informal way of denoting the family $\mathcal{L}$. Suppose further T is connected. If $L_{t_0} \in J(A)$ for some $t_0 \in T(k)$, then $L_t \in J(A)$ $\forall\, t \in T(k)$.

2. We showed (last time) that if $L \in J(A)$ then on $A \times_k A$ we have

$$m^* L \simeq p_1^* L \otimes p_2^* L.$$

(Here $m : A \times_k A \longrightarrow A$ is the group operation)

3. We showed that if D is an effective divisor on A then $2D$ is base point free.

Lemma 1: Let X be a k-scheme, $f, g : X \longrightarrow A$ two morphisms in $Sch/_k$, and L a homogeneous line bundle. Then on $A \times_k A$ we have:

$$(f+g)^* L \simeq f^* L \otimes g^* L.$$

Proof:

Consider the map $(f, g) : X \longrightarrow A \times_k A$ and pull back the relation $m^* L \simeq p_1^* L \otimes p_2^* L$ to X. //

Lemma 2: Suppose L is homogeneous line bundle and $H^0(A, L) \neq 0$. Then L is a trivial bundle, i.e., $L \simeq \mathcal{O}_A$.

Proof: Since $H^0(A, L) \neq 0$, therefore $L \simeq \mathcal{O}(D)$ for some effective divisor D. We have

- $1_X : A \longrightarrow A$, the identity map
- $-1_X : A \longrightarrow A$, the map $a \longmapsto -a$.

Know

$$(1_X + (-1_X))^* L \simeq (1_X^* L) \otimes (-1_X)^* L.$$

i.e. $$\mathcal{O}_A \simeq L \otimes (-1_X)^* L.$$

$$\simeq \mathcal{O}(D) \otimes \mathcal{O}((-1_X)^* D)$$

Since $-1_X : A \longrightarrow A$ is an isomorphism, therefore $(-1_X)^* D$ is effective.

Hence $D + (-1_X)^* D \equiv 0$

$\Rightarrow$ $D = 0$.

Hence $L \simeq \mathcal{O}(D) = \mathcal{O}_A$. //

Lemma 3: Let L be a line bundle on A and

$$\phi_L : A(k) \longrightarrow J(A)$$

the map $a \longmapsto t_a^* L \otimes L^{-1}$. Then ϕ_L is a homomorphism of groups.

Proof:

Know $t_{a+b}^* L \otimes L \simeq t_a^* L \otimes t_b^* L$.

So $t_{a+b}^* \otimes L^{-1} \simeq t_a^* L \otimes t_b^* L \otimes L^{-1} \otimes L^{-1}$

$\Rightarrow$ $\phi_L(a+b) \simeq \phi_L(a) \otimes \phi_L(b)$ //.

Definition: The fixed subgroup of A associated to L is $K(L) := \ker \varphi_L$.

Provisional definition.

Consider the line bundle

$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$

on $A \times A$. $\Lambda(L)\big|_{\{a\}\times A} \simeq t_a^* L \otimes L^{-1}$.

So $\Lambda(L)$ is a family on line bundles on A parametrised by A. On $\{0\}\times A$, $\Lambda(L) \simeq \mathcal{O}_A$.

Know:

$$\{a \in A \mid \Lambda(L)\big|_{\{a\}\times A} \text{ is trivial}\}$$

is closed. In other words $K(L)$ is closed.

We regard $K(L)$ as a finite type k-scheme by giving it its reduced structure.

Note $K(L)^0$ is an abelian subvariety of A, where G^0 is the connected component of the identity for any group scheme G.

Lemma 4: Let L be a line bundle on A. Then $L|_{K(L)^0}$ is a homogeneous line bundle on $K(L)^0$.

Proof: Let $x \in K(L)^0$. Then $x \in K(L)$ and hence $t_x^* L \simeq L$. Note that

$$\begin{tikzcd} A \arrow[r, "t_x"] & A \\ K(L)^0 \arrow[u, hook] \arrow[r, "t_x"'] & K(L)^0 \arrow[u, hook] \end{tikzcd}$$

Hence $\left(t_x^* L\right)|_{K(L)^0} \simeq t_x^*\left(L|_{K(L)^0}\right)$
$$\begin{array}{c} \| \\ L|_{K(L)^0}. \end{array}$$
//

Theorem: If $H^0(A,L) \neq 0$ then $L|_{K(L)}$ is trivial.

Proof: It is enough to show that $L|_{K(L)^0}$ is trivial (why?) Let $Y = K(L)^0$. Then $L|_Y \in \mathcal{J}(Y)$. Let $s \in H^0(A,L)$ be a non-zero section and D the zero locus of s. Then $L \simeq \mathcal{O}(D)$. Since $H^0(A,L) \neq 0$, if $Y = A$, then from what we proved earlier, L is trivial and and we are done. Now assume $Y \neq A$. Then we can always find $a \in A$ such that $D - a \not\subset Y$, i.e. $t_a^* \not\subset Y$. Now we have the easily verifiable:

Fact: $K(t_a^* L) = K(L) \quad \forall a \in A$ (easy!)

Since $Y \not\subset t_a^* D$, it follows that $t_a^* s$ is not everywhere vanishing on Y. Thus $t_a^* s|_Y$ is a non-zero section of $t_a^* L|_Y$. From the fact above, $t_a^* L|_Y$ is a homogenous line bundle on the abelian variety $Y = K(L)^0$. From what we proved earlier today, it follows that $t_a^* L|_Y$ is trivial. This is true for every $a \in A$ such that $Y \not\subset D - a$. Such elements in A form a non-empty open subset U in A (since $Y \neq A$).

Thus $t_a^* L|_Y$ is trivial for $a \in U$. Now $\{t_a^* L|_Y\}_{a \in A}$ is a family of homogeneous bundles on Y parametrised by A (why is it a family?). On the other hand the set of points on A where the members of the family are trivial on Y is a closed set. Since U is non-empty and open in Y, it is dense in Y. It follows that $t_a^* L|_Y$ is trivial for all $a \in A$. Taking $a = 0$ we get $L|_Y$ is trivial. //

[Hint: Consider $(p_1 + p_2)^* L \big|_{Y \times_k A}$.]