Jan 14, 2021.

Lecture 3

For today's lecture we work over an alg. closed field k. A variety, abelian variety, scheme etc will mean objects over k.

Theorem: Let X and Y be complete varieties, Z a connected scheme and $\mathcal{L}$ a line bundle on $X \times Y \times Z$ such that $\mathcal{L}$ restricted to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$, $X \times Y \times \{z_0\}$ is trivial for some $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. Then $\mathcal{L}$ is trivial.

Proof: Later in the course.

Proposition: Let X be a complete variety Z an arbitrary variety. Let $\mathcal{L}$ be a line bundle on $X \times Z$ such that $\mathcal{L}|_{X \times \{z\}}$ is trivial $\forall z \in Z$. Then $\mathcal{L} = p_2^* \mathcal{M}$ for some line bundle $\mathcal{M}$ on Z.

Proof: Later. //

Let A be an abelian variety. Let $p_1, p_2, p_3 : A \times A \times A \longrightarrow A$ be the three projections. Let

$m_{ij} : A \times A \times A \longrightarrow A$ and $m : A \times A \times A \longrightarrow A$

be the maps

$$m_{ij} = p_i + p_j, \qquad m = p_1 + p_2 + p_3.$$

Not a good notation - m is usually reserved for multiplication map.

Let $\mathcal{L}$ be a line bundle on A. Define

$$\mathcal{M} = m^*\mathcal{L} \otimes m_{12}^*\mathcal{L}^{-1} \otimes m_{13}^*\mathcal{L}^{-1} \otimes m_{23}^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}$$

<u>Claim</u>: $\mathcal{M}$ is trivial.

Pf: Enough to show $\mathcal{M}\big|_{\{0\}\times A\times A}$, $\mathcal{M}\big|_{A\times\{0\}\times A}$, $\mathcal{M}\big|_{A\times A\times\{0\}}$

are trivial. By symmetry, it is enough to check that $\mathcal{M}\big|_{\{0\}\times A\times A}$ is trivial.

Let $Z = \{0\}\times A\times A$.

$$m^*\mathcal{L}\big|_Z = (p_2+p_3)^*\mathcal{L}\big|_Z = m_{23}^*\mathcal{L}\big|_Z$$

$$m_{12}^*\mathcal{L}\big|_Z = p_2^*\mathcal{L}\big|_Z$$

$$m_{13}^*\mathcal{L}\big|_Z = p_3^*\mathcal{L}\big|_Z .$$

From this it is clear that $\mathcal{M}\big|_Z$ is trivial. q.e.d.

<u>Remark</u>: Let $\mathcal{L}_0$ = fibre of $\mathcal{L}$ over $0 \in A$. There is a canonical isomorphism,

$$\mathcal{M} \xrightarrow{\sim} \underbrace{\mathcal{L}_0 \otimes_k \mathcal{O}_{A\times A\times A}}_{\pi^*\mathcal{L}_0}$$

where $\pi : A^3 \to \operatorname{Spec} k$ is the structure map.

<u>Proposition</u>: $\mathcal{M}$ is trivial on $A\times A\times A$.

$\longrightarrow$ P.T.O.

Corollary: Let X be a variety, f, g, h maps from X to A. Then

$$(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.$$

Proof: Pull back $\mathcal{M}$ using the map

$$(f, g, h) : X \longrightarrow A \times A \times A.$$ q.e.d.

Notations and definitions: Let A be an abelian variety.

For $a \in A$, define

$$t_a : A \longrightarrow A$$

by $x \longmapsto a + x$.

A line bundle $\mathcal{L}$ on A is called homogeneous if

$$t_a^* \mathcal{L} \simeq \mathcal{L} \qquad \forall\, a \in A.$$

Clearly isomorphism classes of homogeneous line bundles form a group under $\otimes$.

$\mathrm{Pic}(A)$ = isomorphism classes of line bundles on A

$\mathrm{Pic}^0(A)$ = isomorphism classes of line bundles on A alg. equivalent to the trivial line, i.e. $[\mathcal{L}] \in \mathrm{Pic}^0(A) \iff \exists$ connected finite type scheme S and a line bdle $\tilde{\mathcal{L}}$ on $A \times S$ and two points $s_1, s_2 \in S$ with $\tilde{\mathcal{L}}|_{A \times \{s_1\}} \cong \mathcal{O}_A$ and $\tilde{\mathcal{L}}|_{A \times \{s_2\}} \cong \mathcal{L}$.

We will see later in the course that these are the same.

$J(A)$ = isomorphism classes of homog. bundles

Lemma: For $x, y \in A$ and a line bundle $\mathcal{L}$ on A, we have

$$t_{x+y}^* \mathcal{L} \otimes \mathcal{L} = t_x^* \mathcal{L} \otimes t_y^* \mathcal{L}.$$

Proof:

In the corollary take $f \equiv x$, $g \equiv y$, $h = 1_A$. //

For $\mathcal{L} \in \mathrm{Pic}(A)$, define

$$\phi_{\mathcal{L}}(x) = t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \qquad \forall\ x \in A.$$

It is easy to see that $t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ is homogeneous. Indeed,

$$\begin{aligned} t_a^* (t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}) &= t_{a+x}^* \mathcal{L} \otimes t_a^* \mathcal{L}^{-1} \\ &= t_{a+x}^* \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}^{-1} \otimes t_a^* \mathcal{L}^{-1} \\ &= t_a^* \mathcal{L} \otimes t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \otimes t_a^* \mathcal{L}^{-1} \\ &= t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}, \end{aligned}$$

for all $a \in A$. This shows,

Lemma: $\phi_{\mathcal{L}}$ takes values in $J(A)$. ← will use additive notation here.

Lemma: $\phi_{\mathcal{L} \otimes \mathcal{M}} = \phi_{\mathcal{L}} + \phi_{\mathcal{M}}$

Proof: Left to you (Easy!) /

Proposition: Let $\mathcal{L}$ be a line bundle on $A \times S$, where S is variety. For $s \in S$, write L_s for $\mathcal{L}|_{A \times \{s\}}$. Suppose $L_{s_0} \in J(A)$ for some $s_0 \in S$. Then $L_s \in J(A)$ for all $s \in S$.

Proof:

Consider the line bdle $\mathcal{M}$ on $A \times A \times S$ given by

$$\mathcal{M} = (p_1 + p_2)^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_3^* \mathcal{L}^{-1}.$$

We need the following lemma.

Lemma: Let $L \in J(A)$. Let p_1, p_2 be the projections $A \times A \longrightarrow A$ and $m : A \times A \longrightarrow A$ the group operation. Then

$$m^* L \cong p_1^* L \otimes p_2^* L.$$

Proof:

Consider $m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ on $A \times A$. It is trivial on $A \times \{a\}$ $\forall\, a \in A$. We are using homogeneity here. More precisely

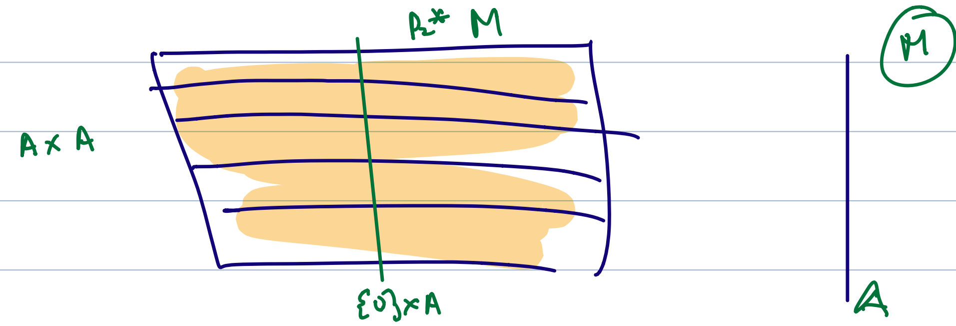
$$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \Big|_{A \times \{a\}} = t_a^* L \otimes L^{-1} \otimes \mathcal{O}_A$$
$$= t_a^* L \otimes L^{-1}$$
$$= \mathcal{O}_A \text{ since } L \text{ is homog.}$$

By an earlier lemma stated today

$$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \simeq p_2^* M$$

for some line bundle M on A.

By symmetry, $m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \Big|_{\{0\} \times A}$ is also trivial. But this is the same as M.



Hence M is trivial. Thus

$$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

is trivial. Hence the lemma. //

Let us return to the proof the Prop^n.

We were considering

$$\mathcal{M} = m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$$

on $A \times A \times S$.

From the lemma, since L_{s_0} is homogenous, therefore $\mathcal{M}|_{A\times A\times\{s_0\}}$ is trivial. On the other hand it is obvious that $\mathcal{M}|_{\{0\}\times A\times S}$ and $\mathcal{M}|_{A\otimes\{0\}\times S}$ are trivial.

Hence $\mathcal{M}$ is trivial //

Proposition: Let D be an effective divisor on an abelian variety A. Then $2D$ is base point free, i.e. $\bigcap_{D' \in |2D|} D' = \phi$.

Proof: Let $L = \mathcal{O}(D)$. We have

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L.$$

Set $y = -x$. Get

$$L^2 \cong t_x^* L \otimes t_{-x}^* L.$$

So $2D \equiv t_x^* D + t_{-x}^* D$

(here $t_x^* D := x + D$
$t_{-x}^* D = -x + D$)

Given a point $a \in A$ we can find x such that $a \notin t_x^* D$ and $t_{-x}^* D$. This proves the lemma. //

Suppose $L = \mathcal{O}(D)$ where D is an effective divisor. We have have map

$$A \xrightarrow{f} \mathbb{P}^N$$

such that $f^* \mathcal{O}(1) \simeq L^2 = \mathcal{O}(2D)$ since $|2D|$ is base-pt free.

$H^0(\mathcal{O}(2D)) \otimes_k \mathcal{O}_A \longrightarrow \mathcal{O}(2D)$

This is surjective since $|2D|$ is b.p free.