April 15, 2021 Lecture 26

Suppose $S = \operatorname{Spec} B$, $\dim_k B < \infty$ (dim as a k v.s.)

then the map $\pi : \Gamma_S \longrightarrow S$ is an isomorphism.

<u>Reminder</u>: Γ_S is the max'l subscheme of $S \times \hat{A}$ such that $M = p_{23}^*(P) \otimes p_{13}^*(L)^{-1}$ is trivial.

on $S \times \hat{A} \times A$

In all this L is a l.b. on $S \times A$ with $L|_{\{s\} \times A} \in \operatorname{Pic}^0(A)$ where s is the unique closed pt of S. The map π is the composite

$$\Gamma_S \hookrightarrow S \times \hat{A} \xrightarrow{p} S.$$

<u>Remarks</u>:

1. It is not necessary to assume $\dim_k B < \infty$. It is enough to assume B is a k-algebra which is an artin local ring. To see this, let $K = B/m_B$, and $\bar{K}$ an algebraic closure of K. Make the base change $k \longrightarrow \bar{K}$, i.e. $\operatorname{Spec} \bar{K} \longrightarrow \operatorname{Spec} k$.

   Then $B \otimes_k \bar{K}$ is a semi-local ring say

   $B \otimes_k \bar{K} = \prod_{i=1}^{n} B_i$, where each B_i local and

   $B_i / m_i B_i = \bar{K}$ and $\dim_{\bar{K}} B_i < \infty$.

   Replace A by $\bar{A} = A \times_k \operatorname{Spec} \bar{K}$ and work with this

   Check $\Gamma_{\bar{S}}$ is the b.c. of Γ_S. Then

   we have the CD (in fact a fibre square)

$$\begin{tikzcd} \Gamma_{\bar S} \arrow[r] \arrow[d, "\bar\pi"'] \arrow[dr, phantom, "\square"] & \Gamma_S \arrow[d, "\pi"] \\ \bar S \arrow[r] & S \end{tikzcd} \qquad \bar S = \mathrm{Spec}(B \otimes_k \bar k)$$

Since $\bar\pi$ is an isomorphism from our earlier results and since $\bar S \longrightarrow S$ is faithfully flat (indeed $k \longrightarrow \bar k$ is faithfully) the assertion follows.

2. In general (with S any k-scheme, not necessary artin local or finite type), our argument shows that the fibres of $\pi : \Gamma_S \longrightarrow S$ are singleton pts. Indeed, have a cartesian square (for $s \in S$)

$$\begin{tikzcd} \pi^{-1}(\{s\}) = \Gamma_{\{s\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_S \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathcal{O}_{S,s}/m_s) = \{s\} \arrow[r] & S \end{tikzcd}$$

The arrow on the left is an isomorphism & hence $\pi^{-1}(\{s\})$ is a singleton. In particular, since π is proper (=proper $\circ$ (closed immersion)) therefore π must be a finite map. In particular π is affine.

3. From the above it is enough to prove that if $s \in S$, and x is the unique point in $\pi^{-1}(s)$, then
$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{\Gamma_S, x}$ is an isomorphism, where the arrow is induced by π. By faithful flatness of completion and since $\mathcal{O}_{\Gamma_S, x}$ is a f.g. module

over $\mathcal{O}_{S,s}$, it is enough to show

$$\hat{\mathcal{O}}_{S,s} \longrightarrow (\mathcal{O}_{\Gamma_S,x}) \otimes_{\mathcal{O}_{S,s}} \hat{\mathcal{O}}_{S,s}$$

is an isomorphism. For this it is enough to show that

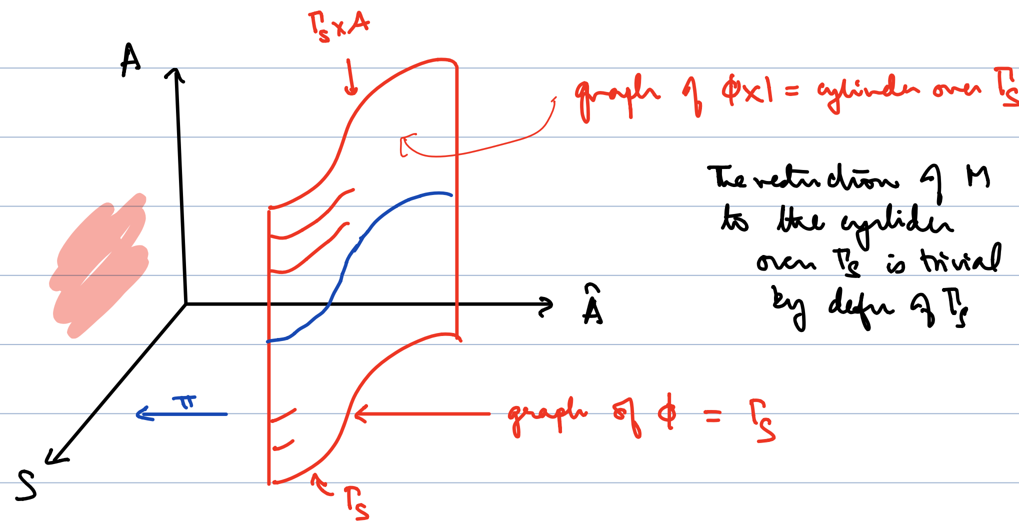
$$\mathcal{O}_{S,s}/\mathfrak{m}_s^n \longrightarrow \frac{\mathcal{O}_{\Gamma_S,x}}{\mathfrak{m}_s^n \mathcal{O}_{\Gamma_S,x}}$$

is an isomorphism $\forall\, n \geq 1$. But we have already proven this since $\mathcal{O}_{S,s}/\mathfrak{m}_s^n$ is an artin local ring.

Conclusion: $\pi : \Gamma_S \longrightarrow S$ is always an isomorphism, for every k-scheme S.

Define $\phi : S \longrightarrow \hat{A}$ as

$$S \xrightarrow{\pi^{-1}} \Gamma_S \subseteq S \times \hat{A} \xrightarrow{p_2} \hat{A}.$$



The restriction of M to the cylinder over Γ_S is trivial by defn of Γ_S

Since $M|_{\Gamma_S \times A}$ is trivial this means

$$p_{23}^*(P)|_{\Gamma_S \times A} \cong p_{13}^*(L)|_{\Gamma_S \times A}.$$

Pulling this back to $S \times A$ $(\phi \times 1)^*$ we get

$$(\phi \times 1)(P) \cong L.$$

We have therefore proved the following:

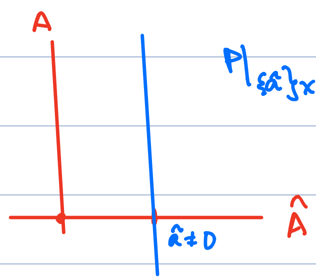
Theorem: Let S be any scheme, and L a line bundle on $S \times A$ such that $L|_{S \times \{0\}}$ is trivial and $L|_{\{s\} \times X} \in \mathrm{Pic}^0(A)$ for every $s \in S$. Then there is a unique morphism $\phi : S \longrightarrow \hat{A}$ such that $L \cong (\phi \times 1)^*(P)$.

Corollary: We have

$$H^i(\hat{A} \times A, P) = \begin{cases} 0 & \text{if } i \neq g \\ k & \text{if } i = g. \end{cases}$$

Proof:

From our earlier computations we know that $R^i p_{1*} P$ is supported at $0 \in \hat{A}$, and hence is artinian $\mathcal{O}_{\hat{A},0}$-module. This is for all i.



$P|_{\{\hat{a}\} \times A} = L_{\hat{a}}$ is a homog. non-trivial l.b. & hence $H^i(L_{\hat{a}}) = 0 \ \forall i$.

So as usual make a base change to $\hat{A}_0 = \mathrm{Spec}(\mathcal{O}_{\hat{A},0})$

Note $\quad H^i(\hat{A}\times A, P) = H^0(\hat{A}, R^i p_{1*}P) = (R^i p_{1*}P)_0$.

Let

$$0 \to F^0 \to \cdots \to F^g \to 0$$

be the Grothendieck complex on $\hat{A}_0$ the pull back of P to $\hat{A}_0 \times A$. Let $R = \mathcal{O}_{\hat{A},0}$.

From our earlier observation

$$H^i(F^\cdot) = 0 \qquad i < g$$

since R is regular local of dim g & $H^i(F^\cdot) \simeq (R^i p_{1*}P)_0$ are artinian modules. With $S = \mathrm{Spec}\, k$, we see that Γ_S is a single point, and hence if $Q = \mathrm{coker}(\check{F}^1 \to \check{F}^0)$, then $Q \simeq k$, for $Q \simeq R/J$, where $\mathrm{Spec}(R/J) = \Gamma_S$, so $J = m_R$, for $\Gamma_S = \{\text{single-pt}\}$. Therefore (using old notations) have an exact seq

$$0 \to Q^{-g} \to Q^{-g+1} \to \cdots \to Q^0 \to Q \to 0$$

where $Q^{-i} := \check{F}^i$. $\qquad Q \simeq k$

So $Q^\cdot$ is a free resolution of k.

Let $x_1, \ldots, x_g \in m_R$ be minimal generators. Since R is a regular local ring, the Koszul complex $\check{K}^\cdot(\underline{x})$ on $x_1, \ldots, x_g$ is also a free resolution of k in the der cat $(Q^\cdot \cong k)$

$$0 \to \check{K}^{-g} \to \cdots \to \check{K}^{-1} \to \check{K}^0 \to k \to 0$$

Dualise $\check{K}^\cdot$. Get another Koszul complex

$$0 \to K^0 \to K^1 \to \cdots \to K^g \to 0 \qquad \text{———} \ (*)$$

This also resolves k, but k sitting in degree g.
($K^\cdot \overset{\text{in the derived cat.}}{\simeq} k[-g]$).

Thus $K^\cdot$ is the Grothendieck complex.

It follows that

$$H^i(A, \mathcal{O}_A) = H^i(\{0\} \times A, P|_{\{0\}\times A})$$

$$= H^i(K^\cdot \otimes_R R/m_R)$$

Now $K^\cdot \otimes_R R/m_R$ has coboundaries which are zero maps, since the coboundaries of $K^\cdot$ are matrices with entries in m_R. It follows that

$$H^i(K^\cdot \otimes_R R/m_R) = K^i \otimes_R R/m_R$$

$$= (\Lambda^i R^g) \otimes_R R/m_R$$

So

$$\dim H^i(A, \mathcal{O}_A) = \binom{g}{i} \qquad \forall i.$$

Also
$$R^i(p_{1*} P)_0 = H^i(K^\cdot) = \begin{cases} 0 & \text{if } i \neq g \\ k & \text{if } i = g \end{cases}$$

Note we have also proved.

Corollary 2 $\dim_k H^i(A, \mathcal{O}_A) = \binom{g}{i}, \quad 0 \le i \le g.$

Remark: Note $H^i(\hat{A}\times A, P) \simeq \bigoplus_{a\in Z} (R^i p_{2*} P)_a$, where $Z \subseteq A$ is the set of pts $a \in A$ s.t. $P|_{\hat{A}\times\{a\}} \simeq \mathcal{O}_{\hat{A}}$.

Setting $i=0$, we see that

$$\cancel{k = H^0(\hat{A}\times A, P) \cong \bigoplus_{a\in Z} H^0(\hat{A}, \mathcal{O}_{\hat{A}})}.$$

$$k = H^g(\hat{A}\times A, P) \cong \bigoplus_{a\in Z} (R^g p_{2*} P)_a$$

It follows that Z is a single point and in fact a reduced point. In other words $Z = \{0\}$.