April 13, 2021 — Lecture 25

k alg. closed field; A an ab. var $/k$, $\dim A = g$.

Let Θ be an ample bundle on A and on $A \times A$ and set $\Lambda = \Lambda(\Theta)$ to be

$$\Lambda(\Theta) = m^*(\Theta) \otimes p_1^*(\Theta)^{-1} \otimes p_2^*(\Theta)^{-1}.$$

$K(\Theta) :=$ max'l subscheme of A over which $\Lambda(L)$ is trivial.

$K(\Theta)$ is a finite subgroup scheme of A.

Set $\hat{A} = A/K(\Theta)$.

We know

$$A(k)/K(\Theta)(k) \xrightarrow{\sim} J(A) \quad \text{via } \phi_\Theta$$
$$a \longmapsto t_a^*\Theta \otimes \Theta^{-1}.$$

Easy to see that the action of $K(\Theta) \times \{0\}$ on $A \times A$ lifts to an action of $K(\Theta)$ on Λ. Therefore Λ descends to a line bundle P on $\hat{A} \times A = \dfrac{A \times A}{K(\Theta) \times \{0\}}$.

We point out that

$$P|_{\hat{A} \times \{0\}} = \mathcal{O}_{\hat{A}} \quad \text{and} \quad P|_{\{0\} \times A} = \mathcal{O}_A$$

<u>Hence</u>

$$\begin{array}{ccc} A & & A \times A \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle p \times 1} \\ \hat{A} & & \hat{A} \times A \end{array} \qquad \begin{array}{c} \Lambda = (p\times 1)^* P. \\ \\ P \end{array}$$

Let S be a k-scheme and L a line bundle on $S \times A$ such that $L|_{S\times\{0\}}$ is trivial, and s.t. $L|_{\{s\}\times A}$ is in $\mathrm{Pic}^0(A)$ for some s. Assume further that S is connected.

On $S \times \hat{A} \times A$ define

$$M = p_{23}^* P \otimes p_{13}^* (L)^{-1}.$$

$$\begin{tikzcd}
\overset{M}{S\times\hat{A}\times A} \arrow[rr, "p_{13}"] \arrow[dr, "p_{23}"] \arrow[dd, "p_{12}"'] & & \overset{L}{S\times A} \arrow[dr] \arrow[dd] & \\
& \overset{P}{\hat{A}\times A} \arrow[rr, "p_1", "p_2"'] \arrow[dd, "p_1"] & & A \arrow[dd] \\
S\times\hat{A} \arrow[rr, "p_2"] \arrow[dr] & & S \arrow[dr] & \\
& \hat{A} \arrow[rr] & & \mathrm{Spec}\, k
\end{tikzcd}$$

$M = p_{23}^* P \otimes p_{13}^* L^{-1}$.

Let

Γ_S = max'l subscheme of $S\times\hat{A}$ on which M is trivial.

Let $\pi : \Gamma_S \longrightarrow S$ be the composite

$$\Gamma_S \subseteq S\times\hat{A} \xrightarrow{p_1} S.$$

(Note, if π is an iso, have $\phi : S \longrightarrow \hat{A}$, namely the composite $S \xrightarrow{\pi^{-1}} \Gamma_S \subseteq S\times\hat{A} \xrightarrow{p_2} \hat{A}$).

We wish to show π is an isomorphism.

We point out that if $u: S' \longrightarrow S$ is a map, then $(u\times 1)^{-1}(\Gamma_S) =$ max'l ~~closed~~ subschemes of $S'\times \hat{A}$ over which $(u\times 1)^* M$ is a trivial family.

$$ (u\times 1)^* M = p_{23}^* P \otimes p_{13}^* L^{-1}. $$

in short form

$$\Gamma_{S'} = (u\times 1)^{-1}\Gamma_S.$$

Therefore if $s \in S$ is a closed point, then with $S' = \operatorname{Spec} k(s) = \{s\}$, we see that $\Gamma_{S'}$ is the set of points $(s, \hat{a}) \in \{s\}\times \hat{A} = \hat{A}$ over which M is trivial. It is not hard to see that $\Gamma_{S'}$ in this case is a reduced, namely the point $(s, \hat{a})$ such that $P|_{\{\hat{a}\}\times A}$ is the line bundle $L|_{\{s\}\times A}$.

This shows, by Nakayama, that the natural map $\mathcal{O}_{\Gamma_S} \otimes k(s,\hat{a}) \longleftarrow \mathcal{O}_{S,s}$ is surjective.

<u>Case 1</u>: Assume $S = \operatorname{Spec} B$ where $\dim_k B < \infty$, i.e., B is an artin local ring. Let s be the only point in S, so that $B = \mathcal{O}_{S,s}$. We also assume WLOG

$$L|_{\{s\}\times A} = \mathcal{O}_A.$$

As we showed above, if $S' = \{s\}$, then $\Gamma_{S'}$ is the locus in $\{s\}\times \hat{A}\times A = \hat{A}\times A$ on which $M|_{\{s\}\times\hat{A}\times A}$ is trivial. Note $M|_{\{s\}\times\hat{A}\times A} = p_{13}^*(L|_{\{s\}\times A})^{-1} \otimes P$.

i.e. $\quad M|_{\{s\}\times\hat{A}\times A} = P\otimes\mathcal{O}_A = P.$

Since, by construction of $\hat{A}$ and P, the only point in A s.t. $P|_{\hat{A}\times\{a\}}$ is trivial is $0\in\hat{A}$, therefore $\Gamma_{S'} = \{s\}\times\{0\}\in S\times\hat{A}$, i.e. $\Gamma_{S'} = \{(s,0)\}$.

This means Γ_S is supported in $(s,0)$, i.e. it is a thickening of $\Gamma_{S'}$.

$$
\begin{tikzcd}
 & \Gamma_{S'} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \Gamma_S \arrow[d, hook] \\
\{s\}\times\hat{A} = & S'\times\hat{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S\times A \arrow[d] \\
\mathrm{Spec}(k(s)) = & S' \arrow[r] & S = \mathrm{Spec}\,B
\end{tikzcd}
\qquad k(s) = B/\mathfrak{m}_B.
$$

Thus Γ_S is an affine scheme, and the nat'l map $B\longrightarrow H^0(\Gamma_S, \mathcal{O}_{\Gamma_S})$, on tensoring over B with $k(s)$ is an isomorphism. So by Nakayama

$$\boxed{B\longrightarrow H^0(\Gamma_S, \mathcal{O}_{\Gamma_S}) \text{ is surjective.}}$$

$$
\begin{tikzcd}
\overset{M}{S\times\hat{A}\times A} \arrow[rr, "p_{13}"] \arrow[dr, "p_{23}"] \arrow[dd, "p_{12}"'] & & \overset{L}{S\times A} \arrow[dr] \arrow[dd] & \\
 & \overset{P}{\hat{A}\times A} \arrow[rr, "p_2"] \arrow[dd, "p_1"] & & A \arrow[dd] \\
S\times\hat{A} \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
 & \hat{A} \arrow[rr] & & \mathrm{Spec}\,k
\end{tikzcd}
$$

Consider

$$R^i_{p_{13,*}} M = R^i_{p_{13,*}} (p_{23}^* P) \otimes L^{-1} \quad \text{(proj. formula)}$$

The locus in $S \times A$ over which M is trivial is finite for the following reason.

$$\begin{tikzcd}
p_{23}^*(\Lambda) & S \times A \times A \arrow[r, "p_{13}"] \arrow[d, "1 \times p \times 1"'] & S \times A \arrow[d, equal] & \\
p_{23}^*(P) & S \times \hat{A} \times A \arrow[r, "p_{13}"'] & S \times A & (s,a)
\end{tikzcd}$$

$$M\big|_{p_{13}^{-1}(s,a)} = P\big|_{\hat{A} \times \{a\}}$$

So $\quad p_{23}^*(\Lambda)\big|_{p_{13}^{-1}(s,a)} =$ inv. image of $P|_{\hat{A} \times \{a\}}$ on $A \times \{a\}$.

$$= t_a^* \Theta \otimes \Theta^{-1}.$$

So if $M|_{p_{13}^{-1}(s,a)}$ is trivial, then $t_a^* \Theta \otimes \Theta^{-1}$ is trivial. This means $a \in K(\Theta)$ which is finite.

<u>Conclusion</u>: The locus in $S \times A$ over which M is trivial is contained in $K(\Theta)$ and hence is finite. Let us write Z for this locus.

If $(s,a) \notin Z$, then it follows that $M|_{\{s\} \times \hat{A} \times \{a\}}$ is a non-trivial line bundle on $\hat{A}$.

Moreover, when $a = 0$, $M|_{\{s\} \times \hat{A} \times \{0\}} = \mathcal{O}_{\hat{A}}$, hence all members in the family of l.b.'s on $\hat{A}$ represented by M are in $\mathrm{Pic}^0(\hat{A})$.

So it follows that if $(s,a) \notin Z$, then

$$H^i\left(\{s\}\times\hat{A}\times\{a\}, M|_{\{s\}\times\hat{A}\times\{a\}}\right) = 0 \qquad \forall i.$$

This means

$$R^i p_{13,*} M$$

is supported on Z $\forall$ i. Since Z is finite, therefore

$$H^q(S\times A, R^i p_{13,*} M) = 0 \qquad \text{for } q \geq 1.$$

Hence by the Leray spectral sequence:

$$H^i(S\times\hat{A}\times A, M) \simeq \Gamma(S\times A, R^i p_{13,*} M).$$

Also, since $L|_Z$ is trivial (all line bundles on finite schemes are trivial), therefore

$$R^i p_{13,*} M = R^i p_{13,*} p_{23}^* P \otimes L^{-1} \simeq R^i p_{13,*} p_{23}^* P.$$

The same argument as above then shows:

$$H^i(S\times\hat{A}\times A, M) \simeq H^i(S\times\hat{A}\times A, p_{23}^* P)$$

$$= B\otimes_k H^i(\hat{A}\times A, P)$$

by flat base change.

In particular $H^i(S\times\hat{A}\times A, M)$ is free f.g. B-module.

Now consider $R^i p_{12,*} M$.

Consider

$$\begin{tikzcd}[column sep=small]
 & \overset{M}{S\times\hat{A}\times A} \arrow[rr, "p_{13}"] \arrow[dr, "p_{23}"] \arrow[dd, "p_{12}"'] & & \overset{L}{S\times A} \arrow[dr] \arrow[dd] & \\
 & & \overset{P}{\hat{A}\times A} \arrow[rr, "p_2"] \arrow[dd, "p_1"] & & A \arrow[dd] \\
R^i p_{12,*} M & S\times\hat{A} \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
 & & \hat{A} \arrow[rr] & & \operatorname{Spec} k
\end{tikzcd}$$

Again M is a family of l.b's which are alg. trivial i.e. $M|_{\{(s,\hat{a})\}\times A}$ is in $\operatorname{Pic}^0(A)$. There is only point in $\hat{A}$ s.t. $P|_{\{\hat{a}\}\times A}$ is trivial & that is $\hat{a}=0$. Hence

$R^i p_{12,*} M$ is supported on $(s,0)\in S\times\hat{A}$,

and hence arguing as we did before,

$$H^i(S\times\hat{A}\times A, M) \xrightarrow{\sim} \left(R^i p_{12,*} M\right)_{(s,0)}.$$

Let $R=\mathcal{O}_{\hat{A},0}$, $C=\mathcal{O}_{S\times\hat{A},(s,0)} = B\otimes_k R$.

<u>Notations</u>: $\hat{A}_0 = \operatorname{Spec} R$

$\operatorname{Spec} C = S\times\hat{A}_0$.

$$\begin{tikzcd}
M & M \\
S \times \hat{A}_0 \times A \arrow[r] \arrow[d, "p_{12}"'] & S \times \hat{A} \times A \arrow[d, "p_{12}"] \\
\operatorname{Spec} C = S \times \hat{A}_0 \arrow[r] \arrow[d] & S \times \hat{A} \arrow[d] \\
S \arrow[r, equal] & S
\end{tikzcd}$$

Consider the Grothendieck complex for the pull back of M to $S \times \hat{A}_0 \times A$ on C. We have a complex of free C-module

$$0 \to F^0 \to F^1 \to \cdots \to F^g \to 0$$

$$\begin{aligned} H^i(F^\cdot) &= H^i(S \times \hat{A}_0 \times A, M) \\ &= H^i(S \times \hat{A} \times A, M) \\ &= (R^i p_{12,*} M)_{(s,0)} \end{aligned}$$

Since $R^i p_{12*} M$ is supported at $\{(s,0)\}$, therefore it is artinian as a C-module.
Hence $H^i(F^\cdot)$ are artinian C-modules

$$\left.\begin{array}{c} C = B \otimes_k R \\ \uparrow \\ R \end{array}\right\} \text{finite as module since } k \to B \text{ is finite}$$

So $H^i(F^\cdot)$ are artinian R-modules.

By what we proved last time, it follows that

$$H^i(F^\cdot) = 0 \quad \text{for } i < g.$$

Moreover, if $N = H^g(F^\cdot)$ then

$$N = H^g(S\times\hat{A}\times A, M) \simeq B\otimes_k \underset{\substack{\uparrow \\ \text{free f.g. } B\text{-module}}}{H^g(\hat{A}\times A, P)}$$

We have an exact seq.

$$0 \to F^0 \to F^1 \to \cdots \to F^g \to N \to 0$$

Recall the module $Q = \mathrm{coker}(\check{F}^1 \to \check{F}^0)$.

Since C is a local ring, we observed,

$Q \cong C/J$ for some ideal J and

$$\Gamma_J = \mathrm{Spec}(C/J).$$

$$\begin{tikzcd} & \Gamma_J \arrow[dl, hook'] \arrow[dr, hook] & \\ S\times\hat{A}_0 \arrow[rr] & & S\times\hat{A} \end{tikzcd}$$

Let $Q^{-i} = \mathrm{Hom}_C(F^i, C) = \check{F}^i$.

Have complex

$$0 \to Q^{-g} \to Q^{-g+1} \to \cdots \to Q^0 \to 0$$

Note $H^{-i}(Q^\cdot) = \mathrm{Ext}^{g-i}_C(N, C)$.

Hence the cohomologies of $Q^\cdot$ are artinian.

So by earlier result

$$H^{-i}(Q^\cdot) = 0 \qquad -g \le -i < 0.$$

and by construction

$$H^0(Q^\cdot) = Q$$

Hence we have exact seq

$$0 \to Q^{-g} \to \cdots \to Q^0 \longrightarrow \underset{\substack{\cap \\ C/J.}}{Q} \longrightarrow 0.$$

Thus, by dualising yet again,

$$N = H^g(F^\cdot) = \operatorname{Ext}^g_C(C/J, C)$$

This means $JN = 0$.

On the other hand N is a free B-module.

$$\begin{tikzcd}
& C/J & N \\
B\otimes_k R = C \arrow[ur] \arrow[drr] & & \\
& & B\otimes 1 \\
B \arrow[uu] \arrow[urr] \arrow[uuur, bend left=60] & &
\end{tikzcd}$$

Since $JN=0$, this amounts to saying

$$J \cap (B\otimes 1) = 0$$

Hence

$$B \longrightarrow C/J = H^0(\Gamma_S, \mathcal{O}_{\Gamma_S}).$$

is injective.

Thus $\pi : \Gamma_S \longrightarrow S$ is an isomorphism. //