

April 13, 2021

Lecture 25

k alg. closed field; A an ab. var / k , $\dim A = g$.

Let \mathcal{O} be an ample bundle on A and on $A \times A$ and

set $\Lambda = \Lambda(\mathcal{O})$ to be

$$\Lambda(\mathcal{O}) = \pi_1^*(\mathcal{O}) \otimes \pi_2^*(\mathcal{O})^{-1} \otimes \mathcal{O}.$$

$k(\mathcal{O}) :=$ maximal subscheme of A over which $\Lambda(L)$ is trivial.

$k(\mathcal{O})$ is a finite subgroup scheme of A .

$$\text{Set } \hat{A} = A/k(\mathcal{O}).$$

We know

$$A(k)/k(\mathcal{O})(k) \xrightarrow{\sim} J(A) \quad \text{via } \phi_{\mathcal{O}} \\ \alpha \mapsto \alpha^* \mathcal{O} \otimes \mathcal{O}^{-1}.$$

Easy to see that the action of $k(\mathcal{O}) \times \{0\}_y$ on $A \times A$

lifts to an action of $k(\mathcal{O})$ on Λ . Therefore

Λ descends to a line bundle \mathcal{P} on $\hat{A} \times A = \frac{A \times A}{k(\mathcal{O}) \times \{0\}_y}$.

We point out that

$$\mathcal{P}|_{\hat{A} \times \{0\}_y} = \mathcal{O}_{\hat{A}} \quad \text{and} \quad \mathcal{P}|_{\{0\}_y \times A} = \mathcal{O}_A$$

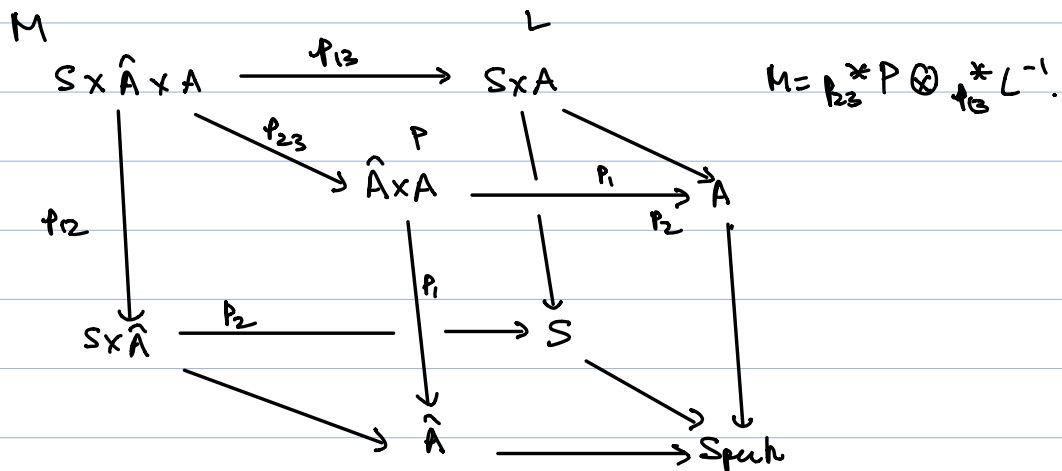
Have

$$\begin{array}{ccc}
 A & & A \times A & & \Lambda = (\pi_1)^* \mathcal{P} \\
 \downarrow \varphi & & \downarrow \pi_1 & & \\
 \hat{A} & & \hat{A} \times A & & \mathcal{P}
 \end{array}$$

Let S be a k -scheme and L a line bundle on $S \times A$ such that $L|_{S \times \{0\}}$ is trivial, and let $L|_{\{s\} \times A}$ is in $\text{Pic}^0(A)$ for some s . Assume further that S is connected.

On $S \times \hat{A} \times A$ define

$$M = p_{23}^* P \otimes p_{13}^* (L)^{-1}.$$



Let

$\Gamma_S = \text{max'l subscheme of } S \times \hat{A} \text{ on which } M \text{ is trivial.}$

Let $\pi : \Gamma_S \rightarrow S$ be the composite

$$\Gamma_S \subseteq S \times \hat{A} \xrightarrow{p_1} S.$$

(Note, if π is an iso, have $\phi : S \rightarrow \hat{A}$, namely the composite $S \xrightarrow{\pi^{-1}} \Gamma_S \subseteq S \times \hat{A} \xrightarrow{p_2} \hat{A}$).

We wish to show π is an isomorphism.

We point out that if $u: S' \rightarrow S$ is a map, then $(u \times 1)^{-1}(\Gamma_S) = \text{maximal closed subspaces of } S' \times \hat{A}$ on which $(u \times 1)^* M$ is a trivial family.

in short form

$$\Gamma_{S'} = (u \times 1)^{-1} \Gamma_S.$$

$$P_{2,2}^* P_{1,2}^* L^{-1}.$$

Therefore if $s \in S$ is a closed point, then with $S' = \text{Spec } k(s) = \{s\}$, we see that $\Gamma_{S'}$ is the set

of points $(s, \hat{a}) \in \{s\} \times \hat{A} = \hat{A}$ over which

M is trivial. It is not hard to see that

$\Gamma_{S'}$ in this case is a reduced, namely the point (s, \hat{a}) such that $P|_{\{s\} \times \hat{A}}$ is the line bundle $L|_{\{s\} \times \hat{A}}$.

This shows, by Nakayama, that the natural map $\mathcal{O}_{\Gamma_S} \otimes k(s, \hat{a}) \leftarrow \mathcal{O}_{S,s}$ is surjective.

Case 1: Assume $S = \text{Spec } B$ where $\dim_k B < \infty$, i.e., B is an artin local ring. Let s be the only point in S , so that $B = \mathcal{O}_{S,s}$. We also assume WLOG

$$L|_{\{s\} \times \hat{A}} = \mathcal{O}_{\hat{A}}.$$

As we showed above, if $S' = \{s\}$, then

$\Gamma_{S'}$ is the locus in $\{s\} \times \hat{A} \times \hat{A} = \hat{A} \times \hat{A}$ on which

$M|_{\{s\} \times \hat{A} \times \hat{A}}$ is trivial. Note $M|_{\{s\} \times \hat{A} \times \hat{A}} = P_{2,2}^* (L|_{\{s\} \times \hat{A}})^{-1} \otimes P_{1,2}^* P_{1,2}^* L^{-1}$.

i.e. $M|_{\{s\} \times \hat{A} \times A} = \mathcal{P} \otimes \mathcal{O}_A = \mathcal{P}$.

Since, by construction of \hat{A} and \mathcal{P} , the only point in A s.t. $\mathcal{P}|_{\hat{A} \times \{s\}}$ is trivial is $0 \in \hat{A}$, therefore $\Gamma_{S'} = \{s\} \times \{0\} \in S \times \hat{A}$, i.e. $\Gamma_{S'} = \{(s, 0)\}$.

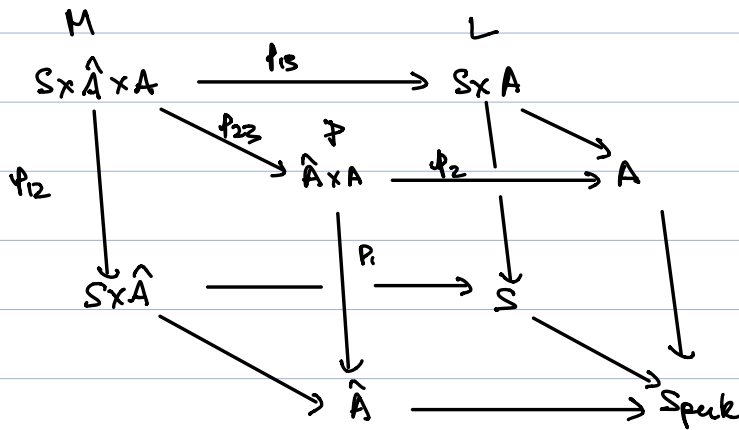
This means Γ_S is supported in $(s, 0)$, i.e. it is a thickening of $\Gamma_{S'}$.

$$\begin{array}{ccc}
 \Gamma_{S'} & \longrightarrow & \Gamma_S \\
 \downarrow & \square & \downarrow \\
 \{s\} \times \hat{A} = S' \times \hat{A} & \longrightarrow & S \times A \\
 \downarrow & \square & \downarrow \\
 \text{Spec}(k(A)) = S' & \longrightarrow & S = \text{Spec } B
 \end{array}$$

$k(A) = B/\mathfrak{m}_B$.

Thus Γ_S is an affine scheme, and the nat'l map $B \rightarrow H^0(\Gamma_S, \mathcal{O}_{\Gamma_S})$, on tensoring over B with $k(A)$ is an isomorphism. So by Nakayama

$B \rightarrow H^0(\Gamma_S, \mathcal{O}_{\Gamma_S})$ is surjective.



Consider

$$R_{P_{13},*}^i M = R_{P_{13},*}^i (P_{22}^* P) \otimes L^{-1} \quad (\text{proj. formula})$$

The locus in $S \times A$ over which M is trivial is finite for the following reason.

$$\begin{array}{ccc}
 P_{23}^*(\Lambda) & S \times A \times A & \xrightarrow{P_{13}} & S \times A \\
 \downarrow \scriptstyle 1 \times p \times 1 & & & \parallel \\
 P_{23}^*(P) & S \times \hat{A} \times A & \xrightarrow{P_{13}} & S \times A \quad (s, a) \\
 M|_{P_{13}^{-1}(s, a)} & = & P|_{\hat{A} \times \{a\}} &
 \end{array}$$

$$\begin{aligned}
 \text{So } P_{23}^*(\Lambda)|_{P_{13}^{-1}(s, a)} &= \text{inv. image of } P|_{\hat{A} \times \{a\}} \\
 &\quad \text{on } \hat{A} \times \{a\}. \\
 &= t_a^* \mathbb{Q} \otimes \mathbb{Q}^{-1}.
 \end{aligned}$$

So if $M|_{P_{13}^{-1}(s, a)}$ is trivial, then $t_a^* \mathbb{Q} \otimes \mathbb{Q}^{-1}$ is trivial. This means $a \in K(\mathbb{Q})$ which is finite.

Conclusion: The locus in $S \times A$ over which M is trivial is contained in $K(\mathbb{Q})$ and hence is finite.

Let us write Z for this locus.

If $(s, a) \notin Z$, then it follows that

$M|_{\{s\} \times \hat{A} \times \{a\}}$ is a non-trivial line bundle on \hat{A} .

Moreover, when $a=0$, $M|_{\{s\} \times \hat{A} \times \{0\}} = \mathbb{Q}_{\hat{A}}$, hence all

members in the family of l.b.'s on \hat{A} represented by M are in $\text{Pic}^0(\hat{A})$.

So it follows that if $(\lambda, a) \notin Z$, then

$$H^i(\mathbb{P}^1 \times \hat{A} \times \mathbb{P}^1, M|_{\mathbb{P}^1 \times \hat{A} \times \mathbb{P}^1}) = 0 \quad \forall i.$$

This means

$$R^i p_{3,*} M$$

is supported on $Z \quad \forall i$. Since Z is finite, therefore

$$H^q(S \times A, R^i p_{3,*} M) = 0 \quad \text{for } q \geq 1.$$

Hence by the long exact sequence:

$$H^i(S \times \hat{A} \times A, M) \cong H^i(S \times A, R^i p_{3,*} M).$$

Also, since $L|_Z$ is trivial (all line bundles on finite schemes are trivial), therefore

$$R^i p_{3,*} M = R^i p_{13,*} p_3^* P \otimes L^{-1} \cong R^i p_{13,*} p_2^* P.$$

The same argument as above then shows:

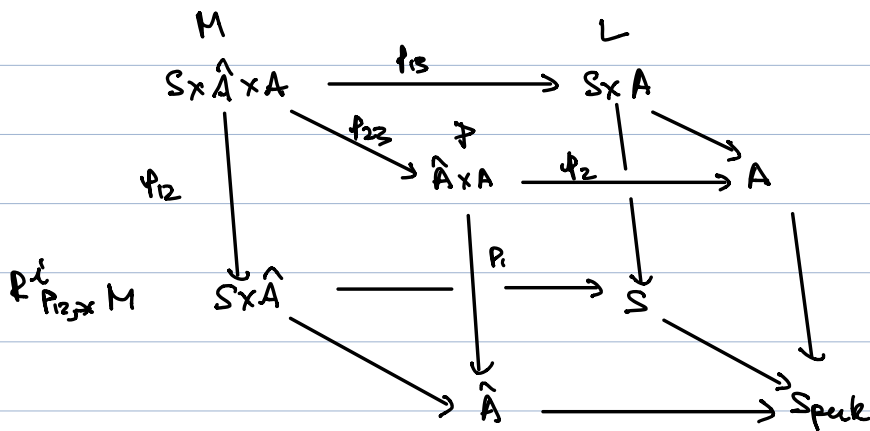
$$\begin{aligned} H^i(S \times \hat{A} \times A, M) &\cong H^i(S \times \hat{A} \times A, p_2^* P) \\ &= B \otimes_k H^i(\hat{A} \times A, P) \end{aligned}$$

by flat base change.

In particular $H^i(S \times \hat{A} \times A, M)$ is free f.g. B -module.

Now consider $R^i p_{12,*} M$.

Consider



Again M is a family of l.b.'s which are alg. trivial
 i.e. $M|_{\{(s, \hat{a})\}_{S \times \hat{A}}}$ is in $\text{Pic}^0(A)$. There is only
 point in \hat{A} s.t. $P|_{\{(s, \hat{a})\}_{S \times \hat{A}}}$ is trivial & that
 is $\hat{a} = 0$. Hence

$R^i p_{23,*} M$ is supported on $(s, 0) \in S \times \hat{A}$,
 and hence arguing as we did before,
 $H^i(S \times \hat{A} \times A, M) \cong (R^i p_{23,*} M)_{(s, 0)}$.

Let $R = \mathcal{O}_{\hat{A}, 0}$, $C = \mathcal{O}_{S \times \hat{A}, (s, 0)} = B \otimes_R P$.

Notations : $\hat{A}_0 = \text{Spc } R$

$\text{Spc } C = S \times \hat{A}_0$.

$$\begin{array}{ccc}
 M & & M \\
 S \times \hat{A}_0 \times A & \longrightarrow & S \times \hat{A} \times A \\
 \downarrow p_{12} & & \downarrow p_{12} \\
 \text{Spec } C = S \times \hat{A}_0 & \longrightarrow & S \times \hat{A} \\
 \downarrow & & \downarrow \\
 S & \xrightarrow{\quad} & S
 \end{array}$$

Consider the Godemick complex for the pull back of M to $S \times \hat{A}_0 \times A$ on C . We have a complex of free C -module

$$0 \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^q \rightarrow 0$$

$$\begin{aligned}
 H^i(F^\bullet) &= H^i(S \times \hat{A}_0 \times A, M) \\
 &= H^i(S \times \hat{A} \times A, M) \\
 &= (R^i p_{12,*} M)_{(S,0)}
 \end{aligned}$$

Since $R^i p_{12,*} M$ is supported at $\{(S,0)\}$, therefore it is artinian as a C -module.

Hence $H^i(F^\bullet)$ are artinian C -modules

$$\begin{array}{c}
 C = B \otimes_k R \\
 \uparrow \\
 R
 \end{array}$$

} finite as module
since $k \rightarrow B$ is
finite

So $H^i(F^\bullet)$ are artinian R -modules.

By what we proved last time, it follows that

$$H^i(F^\bullet) = 0 \quad \text{for } i < g.$$

Moreover, if $N = H^g(F^\bullet)$ then

$$N = H^g(Sx\hat{A} \times_A M) \cong B \otimes_B H^g(A \times_A P)$$

↑
free f.g. B -module

We have an exact seq.

$$0 \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^g \rightarrow N \rightarrow 0$$

Recall the module $Q = \text{coker}(F^1 \rightarrow F^0)$.

Since C is a local ring, we observed,

$$Q \cong C/\mathfrak{J} \quad \text{for some ideal } \mathfrak{J} \text{ and}$$

$$\Gamma_{\mathfrak{J}} = \text{Spec}(C/\mathfrak{J}).$$

$$\begin{array}{ccc} & \Gamma_{\mathfrak{J}} & \\ \curvearrowright & & \curvearrowleft \\ Sx\hat{A}_0 & \longrightarrow & Sx\hat{A} \end{array}$$

$$\text{let } Q^{-i} = \text{Hom}_C(F^i, C) = \check{F}^i.$$

Have complex

$$0 \rightarrow Q^{-g} \rightarrow Q^{-g+1} \rightarrow \dots \rightarrow Q^0 \rightarrow 0$$

$$\text{Note } H^{-i}(Q^\bullet) = \text{Ext}_C^{g-i}(N, C).$$

Hence the cohomologies of Q^\bullet are artinian.

So by earlier result

$$H^{-i}(Q^\bullet) = 0 \quad -g \leq -i < 0.$$

and by construction

$$H^0(Q^\bullet) = Q$$

Hence we have exact seq

$$0 \rightarrow \mathcal{O}^{-g} \rightarrow \dots \rightarrow \mathcal{O}^0 \rightarrow \mathcal{O} \rightarrow 0.$$

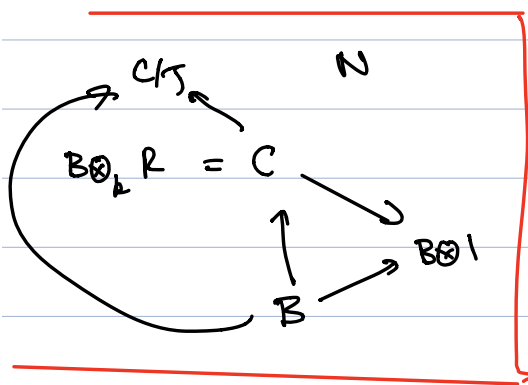
$\mathcal{O} = \mathcal{O}_S.$

Thus, by dualising yet again,

$$N = H^0(F^0) = \text{Ext}_C^g(C/\mathcal{I}, C)$$

This means $JN = 0$.

On the other hand N is a free B -module.



Since $JN=0$, this

amounts to saying

$$J \cap (B \otimes I) = 0$$

Hence

$$B \rightarrow C/\mathcal{I} = H^0(\Gamma_3, \mathcal{O}_{\Gamma_3}).$$

is injective.

Thus $\pi: \Gamma_3 \rightarrow S$ is an isomorphism. //