April 8, 2021

Lecture 24

As usual A is an abelian variety over $k=\bar{k}$, $\dim A = g$.

Recall: If X is complete variety, $S = \operatorname{Spec} B$ an affine k-scheme, and L a line bdle on $X_S = X \times S$. The maximal locus in S over which L is trivial, is a closed subscheme Z of S.

$(1\times\varphi)^* L \cong p_2^* K$ where K is a l.b. on Z'!

$$\begin{tikzcd} X_{Z'} \arrow[r, "1\times\varphi"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X_S \arrow[d] \\ Z' \arrow[r, "\varphi"'] & S \end{tikzcd} \quad L$$

$\Rightarrow$ $Z' \xrightarrow{\varphi} S$ factors as

$$Z' \longrightarrow Z \underset{\text{closed}}{\hookrightarrow} S$$

Let $F^\bullet : 0 \to F^0 \to F^1 \to \cdots \to F^m \to 0$ be "the" Grothendieck complex. Let

$$Q = \operatorname{coker}\left(\check{F}^1 \longrightarrow \check{F}^0\right). \quad Q = Q_B.$$

- For any B-module V
$$H^i(F^\bullet \otimes_B V) \xrightarrow{\simeq} H^i(X_S, L \otimes V)$$

- $\operatorname{Hom}_B(Q, V) \xleftarrow{\sim} H^0(X_S, L \otimes V) \xrightarrow{\sim} H^0(F^\bullet \otimes_B V)$

- If $B \to C$ is a k-algebra map, then $F^\bullet_C := F^\bullet \otimes_B C$ is "the" Grothendieck complex for the pull back of L on $X \times \operatorname{Spec} C$ and $Q_C = Q_B \otimes_B C = Q \otimes C$.

- Let $z \in Y$ be a point such that $L|_{X\times\{z\}}$ is trivial (so that $z \in Z$) then by the above observations

$$\mathrm{Hom}_B\left(Q, \frac{\mathcal{O}_{X,z}}{m_z}\right) \xrightarrow{\sim} H^0(X\times\{z\}, L|_{X\times\{z\}})$$

$$= H^0(X, \mathcal{O}_X)$$

$$= k.$$

Let $k(z) := \mathcal{O}_{X,z}/m_z$. $k \to k(z)$ iso.

Since $k(z) \simeq \mathrm{Hom}_B(Q, k(z)) \simeq \mathrm{Hom}_{k(z)}(Q/m_zQ, k(z))$

therefore Q/m_zQ is 1 dim'l over $k(z)$.

By Nakayama, in a nbhd of z, Q is a cyclic module (i.e. generated by one element.

So locally $Q = B/J$ for some ideal J (locally = around z). We saw that around $z \in Y$. $\mathrm{Spec}(B/J) = Z$.

(*) — $\boxed{\text{In particular, if } B \text{ is local ring then } Z = \mathrm{Spec}\, B/J.}$

<u>Conventions</u>: R local ring then m_R will denote its max'l ideal.

P.T.O. $\longrightarrow$.

Lemma: Suppose R is a regular local ring of dimension g. Let

$$0 \to F^0 \to F^1 \to F^2 \to \cdots \to F^m \to 0$$

be a complex of f.g. free R-modules such that $H^i(F^\cdot)$ are artinian A-modules. Then

$$H^i(F^\cdot) = 0 \quad \text{for } i < g.$$

Proof:

The lemma is obvious if $g=0$.

Now suppose $g>0$. Pick $x \in m_R \setminus m_R^2$.

Then we have a short exact sequence of complexes:

multiplication by x.

$$0 \to F^\bullet \xrightarrow{x} F^\bullet \to \bar{F}^\bullet \to 0$$

where $\bar{F}^i := F^i / xF^i$.

Since $x \in m_R \setminus m_R^2$, therefore $\bar{R} := R/xR$ is also a regular local ring and our induction hypothesis applies to $\bar{R}$. Clearly $\bar{F}^\bullet$ is a complex of f.g. free $\bar{R}$-modules.

We have a long exact sequence

$\longrightarrow$ P.T.O.

$$
\begin{aligned}
0 \to H^0(F^\bullet) &\xrightarrow{x} H^0(F^\bullet) \to H^0(\bar{F}^\bullet) \\
\to H^1(F^\bullet) &\xrightarrow{x} H^1(F^\bullet) \to H^1(\bar{F}^\bullet) \\
\to \cdots & \qquad\qquad \cdots \\
\to H^i(F^\bullet) &\xrightarrow{x} H^i(F^\bullet) \to H^i(\bar{F}^\bullet) \\
\to &
\end{aligned}
$$

It follows that $H^i(\bar{F}^\bullet)$ are artinian R-modules, whence artinian $\bar{R}$-modules.

By induction:

$$H^i(\bar{F}^\bullet) = 0 \quad \text{for } i < g-1.$$

Hence

$$H^{i+1}(F^\bullet) \xrightarrow{x} H^{i+1}(F^\bullet)$$

is injective for $i < g-1$. Since $H^{i+1}(F^\bullet)$ is artinian, it is killed by some power of x, x^m. However $H^{i+1}(F^\bullet) \xrightarrow{x^m} H^{i+1}(F^\bullet)$ is injective for $i < g-1$. So $H^{i+1}(F^\bullet) = 0$ for $i < g-1$. //

Universal property of $\hat{A}$:

Recall $\hat{A} = A/K(\Theta)$ where Θ is a very ample line bundle on A. In greater detail, consider

$$\Lambda = \Lambda(\circledast) = m^*(\circledast) \otimes p_1^* \circledast^{-1} \otimes p_2^* \circledast^{-1}$$

on $A \times A$, then $K(\circledast)$ is the max'l subscheme of A on which Λ is trivial. One checks that $K(\circledast)$ is actually a sub-gp scheme of A.

The Poincare bundle on $\hat{A} \times A$ is the descent of $\circledast$ to $\hat{A} \times A = \dfrac{A \times A}{K(\circledast) \times \{0\}}$.

Let $S = \operatorname{Spec} B$ where B is an artin local ring with $k = B/m_B$. Let s = closed pt of S.

Let L be a line bundle on $S \times A$ such that $L|_{\{s\} \times A} \in \operatorname{Pic}^0(A)$. Suppose further that $L|_{S \times \{0\}}$ is trivial. We would like to prove that $\exists!$ morphism $\phi : S \longrightarrow \hat{A}$ such that $(\phi \times 1_A)^* P \simeq L$.

$$\begin{array}{ccc} L & & P \\ S \times A & \xrightarrow{\phi \times 1_A} & \hat{A} \times A \end{array}$$

Let M be the line bundle on $S \times \hat{A} \times A$ given by

$$M \simeq p_{23}^*(P) \otimes p_{13}^*(L)^{-1}.$$

and let

$$\Gamma_S = \text{max'l closed subscheme of } S \times \hat{A} \text{ on which } M \text{ is trivial.}$$

Let $\pi : \Gamma_S \longrightarrow S$ be the composite $\Gamma_S \subseteq S \times \hat{A} \xrightarrow{p_1} S$. We want to show π is

an isomorphism. Then Γ_s is the graph of a ! map $\phi : S \longrightarrow \hat{A}$, and this ϕ will do the trick.
(ϕ is the composite $S \xrightarrow[p_1^{-1}]{\sim} \Gamma_s \subseteq S \times \hat{A} \xrightarrow{p_2} \hat{A}$).

$$
\begin{tikzcd}
\overset{M}{S \times \hat{A} \times A} \arrow[rr, "p_{13}"] \arrow[dr, "p_{23}"] \arrow[dd, "p_{12}"'] & & \overset{L}{S \times A} \arrow[dr] \arrow[dd] & \\
& \overset{P}{\hat{A} \times A} \arrow[rr, "p_2"] \arrow[dd, "p_1"] & & A \arrow[dd] \\
S \times \hat{A} \arrow[rr, "p_2"] \arrow[dr] & & S \arrow[dr] & \\
& \hat{A} \arrow[rr] & & \operatorname{Spec} k
\end{tikzcd}
$$

$M = p_{23}^* P \otimes p_{13}^* L^{-1}$.

Without loss of generality, we may assume $L|_{\{s\} \times A} = \mathcal{O}_A$. This can be done as follows.

We know $\hat{A}(k) \simeq \mathrm{Pic}^0(A)$, via $\phi_\Theta : A \longrightarrow \mathrm{Pic}(A)$

$a \longmapsto t_a^* \Theta \otimes \Theta^{-1}$

$\ker(\phi_\Theta) = K(\Theta)$

Then $L|_{\{s\} \times A}$ (which belongs to $\mathrm{Pic}^0(A)$) corresponds to a unique point $\hat{a} \in \hat{A}$, and $L|_{\{s\} \times A} \cong P|_{\{\hat{a}\} \times A}$. So replace L by

$L \otimes p_2^* (L|_{\{s\} \times A})^{-1}$.