April 6, 2021

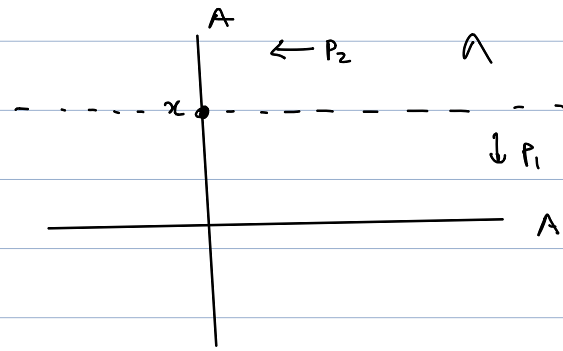
Lecture 23

A = abelian variety over $k = \bar{k}$, $\dim A = g$.

The dual abelian variety :

Let $\Lambda(L)$ be the line bundle on $A \times A$ given by :

$$\Lambda(L) = \Lambda = m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}.$$



Let

$K(L)$ = max'l subscheme of A on which $\Lambda|_{K(L)\times A}$ is trivial.

Proposition: Let

$$\begin{tikzcd} & K(L) \arrow[dr, dashed] & \\ S \arrow[ur, dashed] \arrow[rr, "f"] & & A \end{tikzcd}$$

$f : S \longrightarrow A$ be an S-valued point of A (where $S \in Sch_{/k}$). Set

$$L_S = p_1^* L \quad \text{on} \quad A_S = A \times S.$$

Let $T_f : A_S \longrightarrow A_S$ be the morphism induced by f (this was denoted R_f in HW 5). Then f factors through $K(L)$ if and only if

$$T_f^* L_S \simeq L_S \otimes p_2^* N, \quad N \text{ a l.b. on } S \quad \text{———} (*).$$

Proof :

The following diagram clearly commutes :

Note $T_f(a, s) = (a + f(s), s)$, for T-valued points.

$$\begin{tikzcd}
A_S \arrow[r, "T_f"] \arrow[d, equal] & A_S = A\times S \arrow[d, "p_2"] \\
A\times S \arrow[r, "1\times f"'] & A\times A \arrow[r, "m"'] & A.
\end{tikzcd}$$

Hence

$$T_f^*(L_S) = (1\times f)^* m^* L.$$

This implies.

$$T_f^*(L_S)\big|_{\{0\}\times S} = f^* L$$

Consider the relation (*) :

$T_f^* L_S \simeq L_S \otimes p_2^* N$, N a l.b. on S ——— (*).

Suppose (*) is true for some N. Then

$$T_f^* L_S\big|_{\{0\}\times S} \cong L_S\big|_{\{0\}\times S} \otimes p_2^* N\big|_{\{0\}\times S} \cong N.$$

i.e. $f^* L_S \simeq N$.

Use the fact that $T_f^*(L_S) = (1\times f)^* m^* L_S$ to then conclude that

$$(1\times f)^* \Lambda = (1\times f)^* m^* L \otimes p_1^* L^{-1} \otimes p_2^*(f^* L)^{-1}.$$

Thus (*) holds $\Longleftrightarrow$ $(1\times f)^*\Lambda$ is trivial.

The last is equivalent to saying that f factors through $K(L)$. //

Immediate consequence: $K(L)(S)$ is a subgroup of $A(S)$.

Hence $K(L)$ is a subgroup scheme of A.

Remark Suppose L is ample. Now $K(L)_{red} \subseteq K(L^n)_{red}$, for $n \geq 1$. Since L^n is very ample for $n \gg 0$, it follows that $K(L)_{red}$ is finite. In particular $K(L)$ is a finite group scheme.

We know that if L is ample

$$\phi_L : A \longrightarrow J^0(A) = \mathrm{Pic}^0(A)$$

is surjective. The kernel of ϕ_L can be identified with $K(L)$. Let

$$\hat{A} = A/_{K(L)}.$$

Then from the comments above, morally $\hat{A}$ is a scheme structure on $\mathrm{Pic}^0(A)$.

One can show:

(a) The action of $K(L)$ on $A \times A$ through the first factor lifts to $\Lambda(L)$ and hence $\Lambda(L)$ descends to a line bundle P on $(A\times A)/_{(K(L))}$ $= \left(A/_{K(L)}\right) \times A \simeq \hat{A} \times A$.

P is called the Poincaré bundle on $\hat{A} \times A$.

(b) $$P|_{\{0\} \times A} = \mathcal{O}_A, \quad P|_{\hat{A} \times \{0\}} = \mathcal{O}_{\hat{A}}.$$

(c) Let $S \in Sch/_k$ and suppose $\mathcal{L}$ is a line bundle on $A \times S$ such that $\mathcal{L}|_{A \times \{s\}} \in J^0(A) = \mathrm{Pic}^0(A)$ $\forall s \in S$ then $\exists!$ map $f : S \longrightarrow \hat{A}$ such that

$$(f\times 1)^* P \simeq \mathcal{L} \otimes p_2^* N \quad \text{for some line bdle } N.$$

Remark: Note that $\hat{A}$ is integral, finite type, complete (separatedness?) and hence is an abelian variety. Moreover $A \xrightarrow{\pi_L} \hat{A}$ being finite, we must have

$$\dim \hat{A} = \dim A = g.$$

What are the theorems we wish to prove (if we had another month or so).

1. $$H^i(\hat{A}\times A, P) = \begin{cases} 0 & \text{if } i \neq g \\ k & \text{if } i = g \end{cases}$$

2. $$\dim_k H^i(A, \mathcal{O}_A) = \binom{g}{i}$$

3. If L is a l.b. on A, the set theoretic map $\phi_L : A \longrightarrow \mathrm{Pic}^0(A) \cong \hat{A}$, $a \longmapsto t_a^* L \otimes L^{-1}$ is a group scheme homomorphism and its kernel (as a subgroup scheme) is $K(L)$.

4. $\hat{\hat{A}} \cong A$.

5. Suppose $L \simeq \mathcal{O}(D)$. Then

$$\chi(A, L) = \frac{(D^g)}{g!} \quad \text{(R.R. for abelian varieties)}$$

$$\chi(A, L)^2 = \deg \phi_L .$$

6. Suppose L is non-degenerate, i.e. $K(L)$ is a finite group scheme (e.g L is ample). Then $A \longrightarrow A/_{K(L)}$ is finite and surjective. The map $\phi_L : A \longrightarrow \hat{A}$ is also finite, since the fibres of ϕ_L are cosets of $K(L)$. Since $\dim A = \dim \hat{A}$, this mean ϕ_L is finite & surjective. In particular

$$\hat{A} \simeq A/K(L).$$

7. Let L be non-degenerate. Then $\exists !$ integer i s.t. $H^{i}(A, L) \neq 0$, i.e., for $j \neq i$ $H^{j}(A, L) = 0$. This ! integer is called the index of L and denoted $i(L)$.

8. Suppose L is ample and M is non-degenerate. Let $\Phi(t)$ be the Hilbert poly. of M w.r.t L, i.e.

$$\Phi(n) = \chi(A, M \otimes L^{n}).$$

Then $i(M) = \#$ of +ve roots of Φ.

9. Let L be ample & M any l.b. on A, and Φ the Hilb. poly of M w.r.t. L. Then

$$\Phi(t) = t^{d}\,\psi(t), \quad \psi(0) \neq 0$$

where $d = \dim K(L)$.

10. L ample then L^{n} is very ample for $n \geq 3$.