April 1, 2021 Lecture 22

Let k be an alg. closed field.

Let G be a group scheme over k.

1. An action of G on a finite type scheme X over k is a map

$$\mu : G \times X \longrightarrow X$$

such that

(i) the composite $X = \operatorname{Spec} k \times X \xrightarrow{e \times 1} X \xrightarrow{\mu} X$ is the identity map, where $e : \operatorname{Spec} k \longrightarrow G$ is the identity point (or simply the identity of the group $G(k)$).

(ii) the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times 1} & G \times X \\ \downarrow{\scriptstyle 1 \times \mu} & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow[\mu]{} & X \end{array}$$

commutes.

Equivalently $h_G(S)$ acts on $h_X(S)$ for every $S \in Sch_{/k}$ and this action is functorial.

Equivalently; For

$$S \xrightarrow{x} X$$

an S-valued point of X (i.e. $x \in h_X(S)$).

we have

$$T_x : X_S \longrightarrow X_S$$

such that

(i) $T_x \circ T_y = T_{xy}$ for $x, y \in h_G(S)$

(ii) If $f : S_2 \longrightarrow S_1$ is a map of k-schemes and $x : S_1 \longrightarrow G$ is an S_1-valued point of G then

$$\begin{tikzcd} X \times S_1 \arrow[r, "T_x"] \arrow[d, "1\times f"'] & X \times S_1 \arrow[d, "1\times f"] \\ X \times S_2 \arrow[r, "T_{x(f)}"'] & X \times S_2 \end{tikzcd}$$

where $x(f) = x \circ f : S_2 \longrightarrow G$.

2. Suppose we have a G-action on X, say
$$\mu : G \times X \longrightarrow X.$$
A map $f : X \longrightarrow Y$ in $Sch/_k$ is said to be <u>G-invariant</u> (or <u>simply</u> <u>invariant</u>) if the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

P.T.O $\longrightarrow$

3. Let $\mathcal{F}$ be a coherent sheaf on X. A lift of μ to $\mathcal{F}$ is an isomorphism

$$\lambda : p_2^* \mathcal{F} \longrightarrow \mu^* \mathcal{F}$$

on $G_X = G \times X$ such that on $G \times G \times X$ the diagram below commutes,

$$\begin{tikzcd}
p_3^*(\mathcal{F}) \arrow[rr, "(p_2,p_3)^*(\lambda)"] \arrow[dr, "(m\times 1)^*(\lambda)"'] & & \xi^*(\mathcal{F}) \arrow[dl, "(1\times\mu)^*(\lambda)"] \\
& \eta^*(\mathcal{F}) &
\end{tikzcd}$$

where

$$\xi = \mu \circ (p_2, p_3) \ , \quad \eta = \mu \circ (m \times 1) = \mu \circ (1 \times \mu)$$

The following picture may help with your book-keeping.

$$\begin{tikzcd}
& \lambda : p_2^*\mathcal{F} \to \mu^*\mathcal{F} & & \lambda : p_2^*\mathcal{F} \to \mu^*\mathcal{F} \\
X & G\times X \arrow[l, "\mu"'] & & G\times X \arrow[ddd, "\mu"] \\
& & G\times G\times X \arrow[ul, "(p_2,p_3)"'] \arrow[ur, "m\times 1"] \arrow[ull, bend left, "\xi"] \arrow[d, "1\times\mu"] \arrow[ddr, dashed, "\eta"] & \\
& & G\times X \arrow[dr, "\mu"] & \\
& & \lambda : p_2^*\mathcal{F} \to \mu^*\mathcal{F}. & X
\end{tikzcd}$$

Theorem: (a) Let G be a finite group scheme acting on a scheme X (all schemes are f.t. $/k$) such that the orbit of any point is contained in an affine open subset of X. Then there is a pair (Y,π), where Y is a scheme and $\pi: X \longrightarrow Y$ is a morphism, satisfying the following conditions:

(i) As a top. sp., (Y,π) is the quotient of X for the action of the underlying finite group.

(ii) the morphism $\pi: X \longrightarrow Y$ is G-invariant, and if $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of $\pi_*(\mathcal{O}_X)$ of G-invariant functions, the natural map $\mathcal{O}_Y \longrightarrow \pi_*(\mathcal{O}_X)^G$ is an isomorphism.

The morphism $\pi: X \longrightarrow Y$ is !-ly determined up to isomorphism by these conditions. The morphism π is finite and surjective. Y will be denoted X/G, and it the functional property: Every G-invariant map $f: X \longrightarrow Z$ factors uniquely as

$$X \xrightarrow{\pi} Y \xrightarrow{g} Z.$$

(b) Suppose further that the action of G is free and $G = \mathrm{Spec}(R)$, $n = \dim_k R$. Then π is flat of degree n, i.e. $\pi_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_Y$-module of rank n, and the subscheme of $X \times X$ defined by the closed immersion

error

$$(\mu, p_2) : G\times X \longrightarrow \cancel{G\times X} \quad X\times X$$

is equal to the subscheme $X\times_Y X \subset X\times X$. Finally, if $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module, $\pi^*\mathcal{F}$ has a naturally defined G-action lifting μ, and

$$\mathcal{F} \longmapsto \pi^*\mathcal{F}$$

is an equivalence of the category of $\mathcal{O}_Y$-modules (resp. locally free $\mathcal{O}_Y$-modules of finite rank) and the category of coherent $\mathcal{O}_X$-modules with G-action lifting μ (resp. locally free $\mathcal{O}_X$-modules with G-action lifting μ).

Defn: μ is said to be a free action of G on X if the map

$$G\times X \xrightarrow[(\mu,\, p_2)]{} X\times X$$

is a closed immersion

$$\begin{tikzcd}
G\times_k X \arrow[dr, "\sim"] \arrow[drr, bend left, "\mu"] \arrow[ddr, bend right, "p_2"'] & & \\
& X\times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& X \arrow[r, "\pi"'] & Y = X/G
\end{tikzcd}$$

We have $X\times_Y X \longrightarrow X\times_k X$. Since G is a group scheme, the isomorphism $G\times_k X \xrightarrow{\sim} X\times_Y X$ identifies $X\times_Y X$ as the graph of an equivalence relation on X. In greater detail if $R = X\times_Y X$, then for each $S \in Sch/_k$

$$R(S) \subseteq X(S)\times X(S) = (X\times_k X)(S)$$

is the graph of an equivalence relation on $X(S)$.