

April 1, 2021

Lecture 22

Let  $k$  be an alg. closed field.

Let  $G$  be a group scheme over  $k$ .

1. An action of  $G$  on a finite type scheme  $X$  over  $k$  is a map

$$\mu: G \times X \longrightarrow X$$

such that

(i) the composite  $X = \text{Spec } k \times X \xrightarrow{e \times 1} G \times X \xrightarrow{\mu} X$  is the identity map, where  $e: \text{Spec } k \rightarrow G$  is the identity point (or simply the identity of the group  $G(k)$ ).

(ii) the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times 1} & G \times X \\ \downarrow 1 \times \mu & & \downarrow \mu \\ G \times X & \xrightarrow{\mu} & X \end{array}$$

commutes.

Equivalently  $h_G(S)$  acts on  $h_X(S)$  for every  $S \in \text{Sch}/k$  and this action is functorial.

Equivalently; For

$$S \xrightarrow{x} X$$

an  $S$ -valued point of  $X$  (i.e.  $x \in h_X(S)$ ).

we have

$$T_x: X_S \longrightarrow X_S$$

such that

(i)  $T_x \circ T_y = T_{xy}$  for  $x, y \in h_G(S)$

(ii) If  $f: S_2 \rightarrow S_1$  is a map of  $k$ -schemes and  $x: S_1 \rightarrow G$  is an  $S_1$ -valued point of  $G$  then

$$\begin{array}{ccc} X \times S_1 & \xrightarrow{T_x} & X \times S_1 \\ \downarrow 1 \times f & & \downarrow 1 \times f \\ X \times S_2 & \xrightarrow{T_{x(f)}} & X \times S_2 \end{array}$$

where  $x(f) = x \circ f: S_2 \rightarrow G$ .

2. Suppose we have a  $G$ -action on  $X$ , say  $\mu: G \times X \rightarrow X$ .

A map  $f: X \rightarrow Y$  in  $\mathcal{S}ch/k$  is said to be  $G$ -invariant (or simply invariant) if the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu} & X \\ \downarrow \pi_2 & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

P.T.O  $\rightarrow$

3. Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . A lift of  $\mu$  to  $\mathcal{F}$  is an isomorphism

$$\lambda: p_2^* \mathcal{F} \longrightarrow \mu^* \mathcal{F}$$

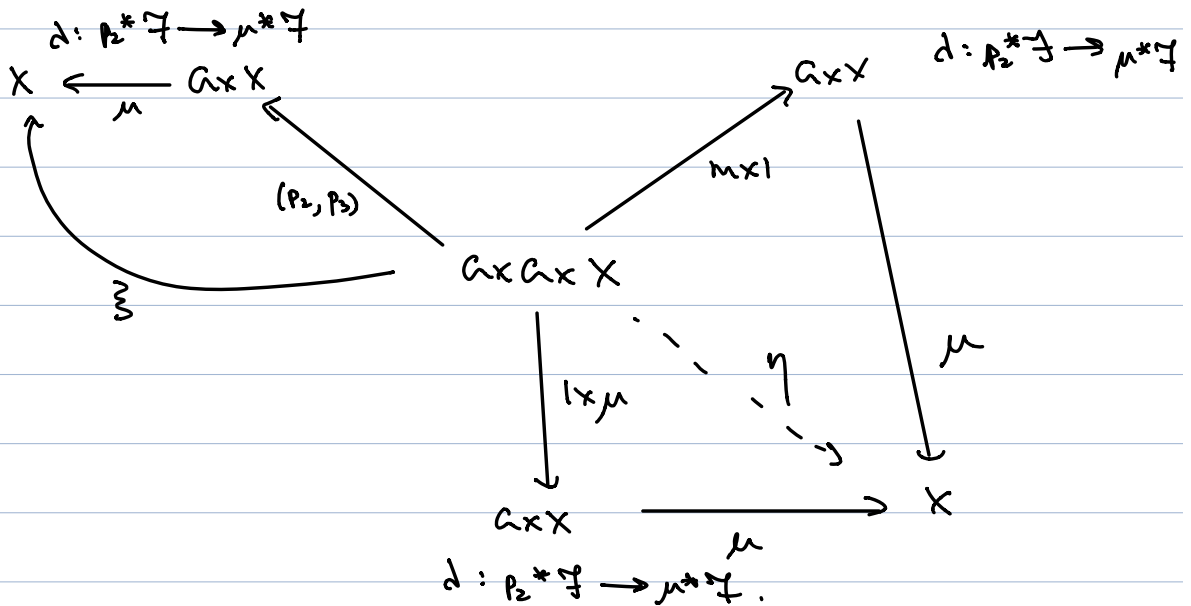
on  $G_X = G \times X$  such that on  $G \times G \times X$  the diagram below commutes,

$$\begin{array}{ccc} p_3^* (\mathcal{F}) & \xrightarrow{(p_2, p_3)^* (\lambda)} & \xi^* (\mathcal{F}) \\ & \searrow (m \times 1)^* (\lambda) & \swarrow (1 \times \mu)^* (\lambda) \\ & \eta^* (\mathcal{F}) & \end{array}$$

where

$$\xi = \mu \circ (p_2, p_3), \quad \eta = \mu \circ (m \times 1) = \mu \circ (1 \times \mu)$$

The following picture may help with your book-keeping.



Theorem: (a) Let  $G$  be a finite group scheme acting on a scheme  $X$  (all schemes are f.t. /  $k$ ) such that the orbit of any point is contained in an affine open subset of  $X$ . Then there is a pair  $(Y, \pi)$ , where  $Y$  is a scheme and  $\pi: X \rightarrow Y$  is a morphism, satisfying the following conditions:

(i) As a top. sp.,  $(Y, \pi)$  is the quotient of  $X$  for the action of the underlying finite group.

(ii) The morphism  $\pi: X \rightarrow Y$  is  $G$ -invariant, and if  $\pi_*(\mathcal{O}_X)^G$  denotes the subsheaf of  $\pi_*(\mathcal{O}_X)$  of  $G$ -invariant functions, the natural map  $\mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)^G$  is an isomorphism.

The morphism  $\pi: X \rightarrow Y$  is  $!$ -by determined up to isomorphism by these conditions. The morphism  $\pi$  is finite and surjective.  $Y$  will be denoted  $X/G$ , and it has the universal property: Every  $G$ -invariant map  $f: X \rightarrow Z$  factors uniquely as

$$X \xrightarrow{\pi} Y \xrightarrow{f} Z.$$

(b) Suppose further that the action of  $G$  is free and  $G = \text{Spec } \mathbb{F}$ ,  $n = \dim_k \mathbb{F}$ . Then  $\pi$  is flat of degree  $n$ , i.e.  $\pi_*(\mathcal{O}_X)$  is a locally free  $\mathcal{O}_Y$ -module of rank  $n$ , and the subscheme of  $X \times X$  defined by

the closed immersion

$$(\mu, p_2): G \times X \longrightarrow \overset{\text{error}}{\cancel{G \times X}} \quad \text{XXX}$$

is equal to the subscheme  $X \times_Y X \subset X \times X$ . Finally, if  $\mathcal{F}$  is a coherent  $\mathcal{O}_Y$ -module,  $\pi^* \mathcal{F}$  has a naturally defined  $G$ -action lifting  $\mu$ , and

$$\mathcal{F} \longmapsto \pi^* \mathcal{F}$$

is an equivalence of the category of  $\mathcal{O}_Y$ -modules (resp. locally free  $\mathcal{O}_Y$ -modules of finite rank) and the category of coherent  $\mathcal{O}_X$ -modules with  $G$ -action lifting  $\mu$  (resp. locally free  $\mathcal{O}_X$ -modules with  $G$ -action lifting  $\mu$ ).

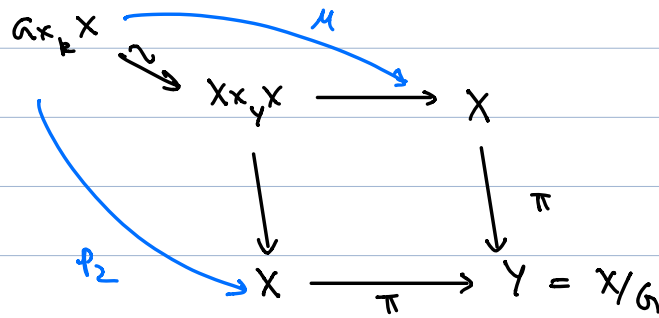
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Defn:  $\mu$  is said to be a free action of

$G$  on  $X$  if the map

$$G \times X \xrightarrow{(\mu, p_2)} X \times X$$

is a closed immersion



We have  $X \times_Y X \longrightarrow X \times_P X$ . Since  $G$  is a group scheme, the isomorphism  $G \times_P X \xrightarrow{\sim} X \times_Y X$  identifies  $X \times_Y X$  as the graph of an equivalence relation on  $X$ . In greater detail if  $P = X \times_Y X$ , then for each  $S \in \text{Sch}/k$

$$P(S) \subseteq X(S) \times X(S) = (X \times_P X)(S)$$

is the graph of an equivalence relation on  $X(S)$ .