March 30, 2021 Lecture 21

<u>Last time</u>: Let X be a complete variety $Y = \mathrm{Spec}\, A$ a scheme, L a line bundle on $X_A = X_Y = X \times_k Y$.

Let $0 \to F^0 \xrightarrow{\delta^0} F^1 \xrightarrow{\delta^1} \cdots$ be the usual Grothendieck complex, $Q = \mathrm{coker}\left((F^1)^\vee \xrightarrow{(\delta^0)^\vee} (F^0)^\vee\right)$.

Know that $p_{2*}(L \otimes_A M) \xrightarrow{\sim} \mathrm{Hom}_A(Q, M)$ for any A-module M. Let W be the locus on which L is trivial, i.e. $y \in Y$ s.t. $L|_{X_y}$ is trivial. Fix a point $y \in W$. Then, shrinking Y around y if necessary, $Q \cong A/J$. Let $Y_1' = \mathrm{Spec}\,(A/J)$. The locus in Y' on which the nat'l map

$$\lambda : p_2^* p_{2*} L|_{X_{Y_1}} \longrightarrow L|_{X_{Y_1}}$$

is an isomorphism, contains the fibre $p_2^{-1}(y)$.

Let Z be the union of the support of $\ker(\lambda)$ and $\mathrm{coker}(\lambda)$. Then Z is closed in $X \times_k Y_1'$, and $p_2^{-1}(y) \cap Z = \phi$. Therefore $p_2(Z)$ is closed subset of Y_1' and does not contain y. Therefore we can find an open subscheme Y_1 of Y_1' such that Y_1 contains y and $L|_{X_{Y_1}} \xrightarrow{\sim} p_2^* M$ for some M (in fact $M = p_{2*}L$). Shrink Y so that Y_1 is closed in Y.

Since we are examining the situation in a nbhd of y, we may assume $M \cong \mathcal{O}_{Y_1}$.

<u>Claim:</u> Suppose $f : Z \longrightarrow Y$ is a morphism of k-schemes such that $\exists$ a line bundle K on Z and an isomorphism $p_2^*(K) \simeq (1 \times f)^*(L)$ on $X \times Z$, then f factors as

$Z \longrightarrow Y_1 \hookrightarrow Y.$

<u>Proof of claim</u>: WLOG can assume $K \simeq \mathcal{O}_Z$.

Also can assume $Z = \operatorname{Spec} B$. We have a map $A \longrightarrow B$ ($Z \to Y$) and we want to prove it factors as $A \twoheadrightarrow A/J \longrightarrow B$.

$$\overset{\shortparallel}{Q}$$

Let $F_B^\bullet = F^\bullet \otimes_A B$, $Q_B = Q \otimes_A B$. It is clear that $F_B^\bullet$ is the Grothendieck complex for $p_2 : X \times_k Z \longrightarrow Z$. Since $\otimes$-product is right exact, $Q_B = \operatorname{coker}(F_1^\vee \otimes_A B \longrightarrow F_0^\vee \otimes_A B)$.

Hence, for any B-module N,

$$\begin{aligned}
\operatorname{Hom}_B(Q \otimes_A B, N) &\simeq p_{2*}(L_Z \otimes_B N). \\
&\simeq p_{2*}(\mathcal{O}_{X \times Z} \otimes_B N) \\
&\simeq p_{2*}(\mathcal{O}_{X \times Z}) \otimes N \quad \text{(proj}^n\text{ formula)} \\
&\simeq \mathcal{O}_Z \otimes N \\
&\cong N.
\end{aligned}$$

In particular, taking $N = B$, get

$$\operatorname{Hom}_B(B/JB, B) \simeq B. \quad (\text{since } Q \cong A/J)$$

The left side is the sub-module of B consisting of element $b \in M$ s.t. $J \cdot b = 0$. Therefore the RS is also killed by elements from J. Thus $JB = 0$. Hence $A \longrightarrow B$ must factor as $A \twoheadrightarrow A/J \longrightarrow B$.

The discussion more or less proves the following Proposition.

Proposition: Let X be a complete variety, Y any scheme and L a line bundle on $X_Y = X \times Y$. Then $\exists !$ closed subscheme $Y_1 \hookrightarrow Y$ having the following properties:

(a) if L_1 is the restriction of L to $X \times Y_1$, there is a l.b. M_1 on L_1 and an isomorphism $p_2^* M_1 \simeq L_1$ on $X \times Y_1$

(b) if $f : Z \longrightarrow Y$ is any morphism s.t. $\exists$ a line bundle K on Z and an isomorphism $(1 \times f)^* L \simeq p_2^*(K)$ on $X \times Z$, then f can be factored as $Z \longrightarrow Y_1 \hookrightarrow Y$.

Proof: If Y_1^* is another closed subscheme with these properties, then Y_1 and Y_1^* are closed subschemes of each other and hence $Y_1^* = Y_1$. So uniqueness is clear.

Suppose Y_1 exists. By the projection formula (or Kunneth), if $p_2^* M_1 \simeq L_1$ then $M_1 \simeq p_{2*} L_1$. Therefore to show $\exists$ M_1 is equivalent to showing $p_{2*}(L_1)$ is an invertible sheaf, and the nat'l map $p_2^* p_{2*}(L_1) \longrightarrow L_1$ is an isomorphism.

In view of these statements, we are reduced to proving that $\exists$ an open cover $\{V_i\}$ of Y s.t. the proposition holds for $X \times V_i \longrightarrow V_i$ and the restriction of L to $X \times V_i$. But we have proved exactly this. //

$\longrightarrow$ P.T.O.

Theorem: Let X and Y be complete varieties, Z a connected k-scheme, and L a line bundle on $X \times Y \times Z$ whose restrictions to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial for some $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$. Then L is trivial.

Proof: Let Z' be the maximal closed subscheme of Z over which L is trivial. Note $z_0 \in Z'$ and hence $Z' \neq \phi$. We have to show $Z' = Z$. Since Z is connected it is enough to show that if a point belongs to Z', Z' contains an open neighbourhood (open subscheme of Z) of that point. Let us denote this point z_0.

Set $\mathfrak{m} = \mathfrak{m}_{z_0}$ - the maximal ideal of $\mathcal{O}_{Z,z_0}$.

Let $I = \mathcal{I}_{z_0}$, where $\mathcal{I}$ is ideal sheaf of Z'.

Note $I \subseteq \mathfrak{m}$.

We have to show that $I = 0$.

By Krull's $\cap$'n theorem

$$\bigcap_{n \geq 0} \mathfrak{m}^n = 0.$$

Suppose $I \neq 0$. We know $I \subseteq \mathfrak{m}$. There exists an integer $n > 0$ such that

$$\mathfrak{m}^n \supset I, \quad \mathfrak{m}^{n+1} \not\supset I.$$

So $$\frac{\mathfrak{m}^{n+1} + I}{\mathfrak{m}^{n+1}} \left(\subset \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \right)$$ is a non-zero k-vector space. Let $J_1 = \mathfrak{m}^{n+1} + I$. We can find J_2 s.t. $\mathfrak{m}^{n+1} \subset J_2 \subset J_1$ such that

$$\dim_k \frac{J_1}{J_2} = 1.$$

So

$$J_1 = J_2 + a\cdot k$$

for some $a \in J_1$. Moreover ~~$J_2 \not\subset I$~~. error. $J_2 \not\supset I$

Let $J_0 = \mathfrak{M}$.

Let Z_i = Closed subscheme of Z consisting of the single point z_0 with structure sheaf $\mathcal{O}_{Z,z_0}/J_i$.

$$\begin{tikzcd}
 & & Z' \arrow[dr, hook] \arrow[d, phantom, "\rotatebox{90}{$\not\subset$}"] & \\
\{z_0\} = Z_0 \arrow[r, hook] & Z_1 \arrow[ur, hook] \arrow[dr, hook] & & Z \\
 & & Z_2 \arrow[ur, hook] &
\end{tikzcd}$$

Let L_0, L_1, L_2 be the restrictions of L to $X\times Y\times Z_0$, $X\times Y\times Z_1$, $X\times Y\times Z_2$

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_0} \xrightarrow{\text{multiplication by } a} \mathcal{O}_{Z_2} \xrightarrow{\text{restriction}} \mathcal{O}_{Z_1} \longrightarrow 0$$

Therefore on the top. space $X\times Y\times Z_0$ we have an exact sequence of sheaves

$$0 \longrightarrow L_0 \xrightarrow{\text{mult. by } a} L_2 \xrightarrow{\text{restriction}} L_1 \longrightarrow 0$$

We know $\mathcal{O}_{X\times Y\times Z_1} \xrightarrow[\lambda]{\sim} L_1$. Therefore we have a nowhere vanishing $s = \lambda(1)$ on L_1. It is easy to see that L_2 is trivial if and only if s can be lifted to a section s' of L_2. Such a section s' is necessarily nowhere vanishing.

Therefore if we can show that $\xi \in H^1(X \times Y \times Z_0, L_0)$ is zero where ξ is the image of s under the connecting map $H^0(X \times Y, L_1) \longrightarrow H^1(X \times Y, L_0)$, then L_2 is trivial.

Since the restrictions of L to $X \times \{y_0\} \times Z$ and $\{x_0\} \times Y \times Z$ are trivial, the restriction of L to $X \times \{y_0\} \times Z_1$ and $\{x_0\} \times Y \times Z_1$ are also trivial, hence the restriction of s to $X \times \{y_0\} \times Z_1$ and $\{x_0\} \times Y \times Z_1$ can be lifted to $L|_{X \times \{y_0\} \times Z_2}$, $L|_{\{x_0\} \times Y \times Z_2}$. This means that the image of ξ by the maps $H^1(X \times Y, \mathcal{O}_{X \times Y}) \longrightarrow H^1(X, \mathcal{O}_X)$ and $H^1(X \times Y, \mathcal{O}_{X \times Y}) \longrightarrow H^1(Y, \mathcal{O}_Y)$ are zero. By the Kunneth formula, $H^1(X \times Y, \mathcal{O}_{X \times Y}) \simeq H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y)$, and hence ξ is zero.

Thus L_2 is trivial, where $Z_2 \subset Z'$ a contradiction. //