March 25, 2021 — Lecture 20

Let X be a complete k-variety, k an algebraically closed field, and let Y be a k-scheme. Let $\mathcal{L}$ be a line bundle on $X\times_k Y = X_Y$.

Aim: To show that the locus on Y on which $\mathcal{L}$ is trivial (i.e. free) has a natural scheme structure. Recall that the locus we are talking about is $y\in Y$ s.t. $\mathcal{L}|_{X\times\{y\}}$ is trivial. We have already seen it is a closed subset of Y. So this locus certainly has the reduced structure, however, as we shall see, there is a more natural scheme structure.

To begin with, we assume $Y = \operatorname{Spec} A$, where A is a k-algebra. We can do this by shrinking Y around a point $y\in Y$ s.t. $\mathcal{L}|_{X\times\{y\}}$ is free. We have the Grothendieck complex

$$F^\bullet: \quad 0\to F^0 \xrightarrow{\delta^0} F^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} F^n \to 0$$

of finite generated free A-modules (projective A-modules) such that for any A-module M

$$H^i(X\times Y, \mathcal{L}\otimes_A M) \xrightarrow{\sim} H^i(F^\bullet\otimes_A M).$$

Recall also that if $Q = \operatorname{coker}\left((F^1)^\vee \xrightarrow{(\delta^0)^t} (F^0)^\vee\right)$, then for any A-module M,

$$H^0(X\times Y, \mathcal{L}\otimes M) \xrightarrow{\sim} \operatorname{Hom}_A(Q, M)$$

Moreover, this isomorphism is functorial in M.

We can say more. Suppose B is an A-algebra. Tensor the Grothendieck complex with B (over A). Get

$$0 \to F^0 \otimes_A B \longrightarrow F^1 \otimes_A B \longrightarrow \cdots \longrightarrow F^n \otimes_A B \longrightarrow 0.$$

Let X_B be the base change $X \times_k \operatorname{Spec} B = X \times_Y \operatorname{Spec} B$.

$$\begin{tikzcd} X_B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{Spec} B \arrow[r] & Y \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Using standard fact

$$H^i(F^\bullet \otimes_A B) \xrightarrow{\sim} H^i(X_B, \mathcal{L}_B)$$

where $\mathcal{L}_B$ is the pull back of $\mathcal{L}$ under $X_B \longrightarrow X_Y$. In particular

$$\Gamma(X_B, \mathcal{L}_B) = H^0(X_B, \mathcal{L}_B) \xrightarrow{\sim} H^0(F^\bullet \otimes_A B) = \operatorname{Hom}_A(Q, B)$$
$$= \operatorname{Hom}_B(Q \otimes_A B, B).$$

Recall we had point $y \in Y$ such that $\mathcal{L}|_{X \times \{y\}}$ is trivial. Let $B = k(y)$ the residue field at y. The above observations show that

$$\Gamma(X, \mathcal{L}|_{X \times \{y\}}) \xrightarrow{\sim} \operatorname{Hom}_A(Q, A/\underline{m})$$

where $\underline{m}$ is the maximal ideal corresponding to y. (Note $k(y) = A/\underline{m}$). Since $\mathcal{L}|_{X \times \{y\}}$ is trivial, so

$$H^0(X, \mathcal{L}|_{X \times \{y\}}) \xrightarrow{\sim} H^0(X, \mathcal{O}_X) = k$$

since X is a complete variety. In particular

$\operatorname{Hom}_A(Q, A/\underline{m}) \xrightarrow{\sim} \Gamma(X, \mathcal{L}|_{X \times \{y\}})$ is one dimensional as a vector space over k.

Now,

$$\mathrm{Hom}_A(Q, A/\underline{m}) \xrightarrow{\sim} \mathrm{Hom}_A(Q/\underline{m}Q, A/\underline{m})$$

$$= \mathrm{Hom}_{A/\underline{m}}(Q/\underline{m}Q, A/\underline{m})$$

So $\dim_k \mathrm{Hom}_A(Q, A/\underline{m}) = \dim_k Q/\underline{m}Q$

Hence

$$\dim_k Q/\underline{m}Q = 1.$$

Let $q \in Q$ be such that $q \notin \mathfrak{m}Q$. Let

$$A \xrightarrow{\phi} Q$$

be the map $a \longmapsto aq$. This is well-defined since A is a free A-module. Moreover, by Nakayama,

$A_m \xrightarrow{\phi_m} Q_{\underline{m}}$ (the localisation map) is surjective

since $\underset{\substack{\| \\ A/\underline{m}}}{A \otimes_A A/m} \longrightarrow Q \otimes A/m = Q/mQ$ is surjective.

Hence in a neighbourhood of y ϕ is surjective. Shrinking Y further, around y, we may assume $A \xrightarrow{\phi} Q$ is surjective. Let $J = \ker \phi$. Then J is an ideal in A, and we can identify Q with A/J.

Let

$$Y_1' = \mathrm{Spec}\, A/J \overset{\text{closed}}{\hookrightarrow} Y.$$

Let $\mathcal{L}_1'$ be the restriction of $\mathcal{L}$ to Y_1'.

$$\begin{tikzcd}[column sep=large]
X\times_k Y_1' = X_Y \times_Y Y_1' = \overset{\mathcal{L}_1'}{X_{Y_1'}} \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & \overset{X_Y\ \mathcal{L}}{\overset{\shortparallel}{X\times_k Y}} \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
Y_1' \arrow[r, hook] & Y \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Setting $B = A/J$ in our computations, we have

$$\begin{aligned}
p_{2*}\mathcal{L}_1' &\cong \operatorname{Hom}_A(Q, B) \\
&= \operatorname{Hom}_A(B, B) \qquad \text{(since } A \xrightarrow{\phi} Q \text{ is surjective, } J = \ker\phi, \ \& \ B = A/J) \\
&\simeq B \\
&= A/J
\end{aligned}$$

More precisely $p_{2*}\mathcal{L}_1'$ is the coherent sheaf associated with the A/J-module A/J.

We have shown, therefore, that $p_{2*}\mathcal{L}_1'$ is a free rank one $\mathcal{O}_{Y_1'}$-module, i.e. it is a line bundle on Y_1'. Consider the natural homomorphism

$$p_2^* p_{2*} L_1' \longrightarrow L_1'.$$

(Recall (p_2^*, p_{2*}) are an adjoint pair: Hence

$$\operatorname{Hom}_{X_{Y_1'}}(p_2^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{Y_1'}(\mathcal{F}, p_{2*}\mathcal{G})$$

Let $\mathcal{F} = p_{2*}\mathcal{G}$ in the above. The right side then is $\operatorname{Hom}_{Y_1'}(p_{2*}\mathcal{G}, p_{2*}\mathcal{G})$ which has a distinguished element, namely the identity. Hence we we get an element in the right side)

We want to prove that the map

$$\lambda : p_2^* p_{2*} \mathcal{L}_1' \longrightarrow \mathcal{L}_1'$$

is an isomorphism. This need not be true.

By Nakayama, and the fact that both are line bundles, this map is an isomorphism at at point $p \in X \times_k Y_1'$ if and only if the induced map

$$(p_2^* p_{2*} \mathcal{L}_1') \otimes k(p) \longrightarrow \mathcal{L}_1' \otimes k(p).$$

is surjective.

If $p \in p_2^{-1}(y)$, where y is our special point, then the above is true. Indeed, we simply note that

$$\begin{array}{ccc} \mathrm{Hom}_A(A/J, A/J) & \longrightarrow & \mathrm{Hom}_A(A/J, A/m) \\ \| & & \| \\ A/J & & A/m \end{array}$$

is surjective. Therefore λ is an isomorphism on all points of $p_2^{-1}(y)$.

Let Z be the union of the support of $\ker \lambda$ and $\mathrm{coker}\, \lambda$. We have seen that $Z \cap p_2^{-1}(y) = \phi$. Since p_2 is a proper map, this means $p_2(Z)$ is closed in Y_1' and does not contain $y \in Y_1'$.

Shrinking $\mathrm{Spec}\, A$ further, we can find $Y_1 = \mathrm{Spec}\, A/J$ on which $\mathcal{L}_1 :=$ pull back of $\mathcal{L}$ to Y_1 is of the form $p_2^* \mathcal{M}$ where $\mathcal{M}$ is a line bundle on Y_1

Using the above discussion, we can prove:

Proposition: Let X be a complete variety, Y any scheme and $\mathcal{L}$ a line bundle on $X \times Y$. Then there exists a unique closed subscheme $Y_1 \hookrightarrow Y$ having the following properties:

(a) If $\mathcal{L}_1$ is the restriction of $\mathcal{L}$ to $X \times Y_1$, there is a line bundle M_1 on Y_1 and an isomorphism $p_2^* M_1 \simeq \mathcal{L}_1$ on $X \times Y_1$.

(b) if $f: Z \longrightarrow Y$ is any morphism such that $\exists$ a line bundle K on Z and an isomorphism $p_2^*(K) \cong (1 \times f)^*(\mathcal{L})$ on $X \times Z$, then f factors as $Z \longrightarrow Y_1 \hookrightarrow Y$.