Jan 7, 2021

Lecture 2

Corollaries to the rigidity lemma (we are working over an alg closed field k).

Cor. 1 : Suppose A and B are abelian varieties, $f: A \longrightarrow B$ a morphism of varieties such that $f(e_A) = e_B$, where e_A is the identity in A and e_B the identity on B. Then f is a group homomorphism.

Remarks : 1. Since we are working over an alg. closed field, we do not distinguish between X and $X(k)$, where X is is a k-variety.

2. Technically $e_A \in X(k)$, $e_B \in Y(k)$.

Proof: Consider

$$h(x_1, x_2) = f(x_1 x_2) f(x_1)^{-1} f(x_2)^{-1}, \quad x_1, x_2 \in A(k).$$

Clearly $h(e_A, x) = e_B \quad \forall\ x \in A(k)$

$h(x, e_A) = e_B \quad \forall\ x \in A(k)$.

By the rigidity lemma, h is a constant, in fact

$$h(x_1, x_2) = e_B \quad \forall\ x \in A(k).$$

The theorem follows.

$\longrightarrow$ PTO

Cor 2 : $A(k)$ is an abelian group

Proof: Let $f: A \longrightarrow A$ be the map $x \longmapsto x^{-1}$ and apply Cor 1. q.e.d.

Cor 3 : Let A be an abelian variety. Regard it as a variety with a base point e (e = identity). Let S and T be complete varieties with base points s_0 and t_0. Then the natural map

$$\mathrm{Hom}(S, A) \times \mathrm{Hom}(T, A) \longrightarrow \mathrm{Hom}(S \times_k T, A)$$

$$(f, g) \longmapsto (x, y) \longmapsto f(x) + g(y)$$

is a bijection.

Remarks: The Hom here is to denote morphisms in the cat. of varieties with base pt. We are using the additive notation for group operations.

Proof: Suppose $f(x) + g(y) = f'(x) + g'(y)$ for some $f, f' \in \mathrm{Hom}(S, A)$, $g, g' \in \mathrm{Hom}(T, A)$ and $\forall (x, y) \in S \times T$

Set $x = s_0$. Get

$$g(y) = g'(y) \quad \forall\ y \in T.$$

Similarly, set $y = t_0$. Get

$$f(x) = f'(x) \quad \forall\ x \in S,$$

hence the given map is injective.

Suppose $h \in \mathrm{Hom}(S \times T, A)$.

Set $f(s) = h(s, t_0)$

$g(t) = h(s_0, t)$.

Consider

$$k(s,t) = h(s,t) - f(s) - g(t).$$

Setting $s = s_0$, get

$$k(s_0, t) = e \qquad e = 0.$$

Setting $t = t_0$, get

$$k(s, t_0) = e$$

Thus, by the rigidity lemma, $k(s,t) \equiv e$. Hence h is the image of (f, g). q.e.d.

Basic sheaf cohomology:

Let X be a top. space, $\mathcal{F}$ a sheaf on X, and

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$$

an injective resolution of $\mathcal{F}$ (i.e. $\mathcal{J}^\bullet$ is an inj. resln of $\mathcal{F}$).

$$H^i(X, \mathcal{F}) := H^i(\Gamma(X, \mathcal{J}^\bullet)) \qquad \forall i \in \{0, 1, \ldots\}.$$

Generalities: Suppose $\mathcal{A}$ is an abelian category with enough injectives (every object is a subobject of an injective object). Suppose $T : \mathcal{A} \to \mathcal{B}$ is an additive left exact functor from $\mathcal{A}$ to another abelian category $\mathcal{B}$.

The right derived functors of T are defined as follows: Let $A \in \mathcal{A}$. Let $I^\cdot$ be an inj. resln of A. Set

$$R^i T(A) = H^i(T(I^\cdot)) \quad , \; i \geq 0.$$

An object $A \in \mathcal{A}$ is said to be T-acyclic if $R^i T(A) = 0 \quad \forall \; i \geq 1$. (Note $R^0 T = T$.)

<u>Easy fact</u>: If

$$0 \to A^0 \to A^1 \to A^2 \to \cdots$$

is an exact of T-acyclic objects, then,

$$0 \to T(A^0) \to T(A^1) \to T(A^2) \to \cdots$$

is an exact sequence. (Hint: Use the fact that if $0 \to A \to B \to C \to 0$ is exact with A T-acyclic, then $0 \to T(A) \to T(B) \to T(C) \to 0$ is exact)

<u>Fact</u>: Let $E^\cdot$ be an A-acyclic resolution of an object $A \in \mathcal{A}$. Let $I^\cdot$ be an inj resln of A. Let

$$\varphi : E^\cdot \to I^\cdot$$

be the homotopy ! map lifting the identity on A.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E^0 \arrow[r] \arrow[d] & E^1 \arrow[r] \arrow[d] & \cdots \arrow[d] \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

Then $T(\varphi) : T(E^\cdot) \longrightarrow T(I^\cdot)$ is a quasi-isomorphism.
In particular

$$H^i(T(\varphi)) : H^i(T(E^\cdot)) \longrightarrow R^i T(A)$$

is an isomorphism.

Proof:

Let $C^\cdot_\varphi$ be the mapping cone of φ ($\varphi : E^\cdot \longrightarrow I^\cdot$). Since φ is a quasi-iso, $C^\cdot_\varphi$ is exact. Now $C^\cdot_\varphi$ consists of $\mathcal{F}$ acyclic. From the "easy part", $T(C^\cdot_\varphi)$ is exact.

Recall $C^n_\varphi = I^{n+1} \oplus E^n$. So $T(C^n_\varphi) = T(I^{n+1}) \oplus T(E^n)$.

It is easy to see, $T(C^\cdot_\varphi) = C^\cdot_{T(\varphi)}$.

Since $C^\cdot_{T(\varphi)}$ is exact, $T(\varphi)$ is a quasi-iso. q.e.d.

Cech complexes:

Let $\mathcal{U}$ be an open cover of the top. space X.
Have Čech complex

$$C^\cdot(\mathcal{U}, \mathcal{F})$$

for any sheaf $\mathcal{F}$ on X.

For an open set $U \subseteq X$, let $\mathcal{U} \cap U$ denote the open cover of U obtained by intersecting the members of $\mathcal{U}$ with U.

Have an assignment

$$U \longmapsto C^\cdot(\mathcal{U} \cap U, \mathcal{F}|_U)$$

This assignment gives us a sheaf of complexes (=

complex of sheaves) $\mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$.

$\mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$ is called the sheaf Čech complex.

Note $\mathscr{C}^\cdot(\mathcal{U}, \mathcal{F}) \neq C^\bullet(\mathcal{U}, \mathcal{F})$.

Fact: The nat'l map $\mathcal{F} \xrightarrow{\varepsilon} \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$ gives a resln

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathcal{U}, \mathcal{F}) \to \mathscr{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

See [Hart, pp. 220-221, Chap III, Lemma 4.2].

Hence we have a homotopy ! quasi-iso

$$\mathscr{C}^\cdot(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{E}^\cdot$$

where $\mathcal{E}^\cdot$ is an injective resolution of $\mathcal{F}$.

Hence have maps, one for each $i \geq 0$

$$\begin{array}{ccc} H^i(\Gamma(X, \mathscr{C}^\cdot(\mathcal{U}, \mathcal{F}))) & \longrightarrow & H^i(\Gamma(X, \mathcal{E}^\cdot)) \\ \| & & \| \\ \check{H}^i(\mathcal{U}, \mathcal{F}) & & H^i(X, \mathcal{F}) \end{array}$$

The result therefore is:

$\exists$ maps, one for each i,

$(*)_i$: $\qquad \check{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$

<u>Schemes</u>: Let X be a separated scheme, $\mathcal{U}$ a cover of X consisting of affine open subschemes of X, and $\mathcal{F}$ a quasi-coherent sheaf. Then the maps $(*)_i$ are isomorphisms for all i.

Kunneth formula: Suppose X and Y are finite type k-schemes, $p_1 : X \times_k Y \longrightarrow X$, $p_2 : X \times_k Y \longrightarrow Y$ the projections, then

$$H^n(X \times_k Y, \mathcal{F} \boxtimes \mathcal{G}) \xrightarrow{\sim} \bigoplus_{i+j=n} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}).$$

Here, $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent on X and Y resp., and $\mathcal{F} \boxtimes \mathcal{G} := p_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} p_2^* \mathcal{G}$.