March 23, 2021 Lecture 19

Summary of where we are

- If A is an abelian variety then $J^0(A)$ is the group of (isomorphism classes of) line bundles L such that $t_x^* L \otimes L^{-1} \simeq \mathcal{O}_A$.
- We showed that L is algebraically equivalent $\mathcal{O}_A$ if and only if $[L] \in J^0(A)$.

Therefore we can identify (and will do so) $J^0(A)$ with $\mathrm{Pic}^0(A)$, where $\mathrm{Pic}^0(A)$ is the isomorphism classes of line bundles algebraically equivalent to $\mathcal{O}_A$.

- We also showed that if $H^0(L) \neq 0$ then L is ample if and only $K(L)$ is finite.
- If $L \in \mathrm{Pic}^0(A)$ and $L \not\simeq \mathcal{O}_A$ then $H^i(L) = 0 \ \forall i$.

Theorem: Let L be ample and $M \in \mathrm{Pic}^0(A)$. Then for some $x \in A$,

$$M \simeq t_x^* L \otimes L^{-1}.$$

Remark: Recall that for each line bundle L we had group homomorphism

$$\phi_L : A \longrightarrow \mathrm{Pic}^0(A)$$
$$x \longmapsto t_x^* \otimes L^{-1}.$$

What theorem says is that if L is ample then ϕ_L is surjective. Moreover, from earlier results, we do know that if L is ample $K(L) = \ker(\phi_L)$ is finite. Hence

$$\mathrm{Pic}^0(A) \simeq A/K(L).$$

Proof:

Let K be the line bundle on $A \times A$ given by

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^*(L^{-1} \otimes M^{-1})$$

Let $x \in A$.

$$p_1^{-1}(x) = \{x\} \times A, \quad p_2^{-1}(x) = A \times \{x\}.$$

Now

$$K|_{p_1^{-1}(x)} = t_x^* L \otimes L^{-1} \otimes M^{-1}.$$

and

$$K|_{p_2^{-1}(x)} = t_x^* L \otimes L^{-1}.$$

Suppose M is not isomorphic to $t_x^* L \otimes L^{-1}$ for any $x \in A$. Then $K|_{p_1^{-1}(x)}$ is non-trivial for every $x \in A$. On the other hand $K|_{p_1^{-1}(x)} \in \mathrm{Pic}^0(A)$ $\forall$ $x \in A$. Therefore

$H^i(K|_{p_1^{-1}(x)}) = 0$ $\forall i$.

Consider

$$A \times A \xrightarrow{p_1} A.$$

From semi-continuity we see that $R^i p_{1*} K = 0$ $\forall i$.

Let $K \longrightarrow \mathcal{E}^\bullet$ be an injective resolution. From the above $p_{1*}\mathcal{E}^\bullet$ is exact. By the right exactness of p_{1*}, we know $0 \to p_{1*}K \to p_{1*}\mathcal{E}^0 \longrightarrow p_{1*}\mathcal{E}^1$ is also exact. It follows that

$$0 \to p_{1*}K \longrightarrow p_{1*}\mathcal{E}^\bullet$$

is exact, and since $p_{1*}\mathcal{E}^\bullet$ is exact this means $p_{1*}K = 0$.

Now
$$\begin{aligned} H^i(A\times A, K) &= H^i(\Gamma(A\times A, \mathcal{E}^\bullet)) \\ &= H^i(\Gamma(A, p_{1*}\mathcal{E}^\bullet)) \\ &= 0 \qquad \forall i. \end{aligned}$$

Now consider $K|_{p_2^{-1}(x)}$. We have observed that $K|_{p_2^{-1}(x)} \cong t_x^* L \otimes L^{-1}$. Therefore, if $x \notin K(L)$, we have
$$H^i(A, K|_{p_2^{-1}(x)}) = 0 \qquad \forall i.$$
Since $K|_{p_2^{-1}(x)} \cong t_x^* L \otimes L^{-1}$ is trivial if and only if $x \in K(L)$ we see that $R^i p_{2*}K$ is supported on $K(L)$.

Using the Grothendieck complex, one then sees that
$$R^i p_{2*}K = \bigoplus_{x\in K(L)} (R^i p_{2*}K)_x$$

Using the Leray spectral sequence, this means
$$H^i(A\times A, K) \simeq \bigoplus_{x\in K(L)} (R^i p_{2*}K)_x. \quad \forall i$$
However, we've seen that the LHS of the above is zero. So the RHS is zero, i.e. $(R^i p_{2*}K)_x = 0$, $\forall i$ & $\forall\, x \in K(L)$. We already know $(R^i p_{2*}K)_x = 0$ $\forall i$, and $x \notin K(L)$, since $H^i(A, K|_{p_2^{-1}(x)}) = 0$ $\forall i$.

Hence $H^i(A, K|_{p_2^{-1}(x)}) = 0$ $\forall i$, and $\forall\, x \in A$.

If we pick $x=0$, we have $K|_{p_2^{-1}(x)} = t_0^* L \otimes L^{-1} = \mathcal{O}_A$, and we know $H^0(A, \mathcal{O}_A) \neq 0$. This is a contradiction. //

Things we'd like to do:

1. Show that L is non-degenerate, i.e. $K(L)$ is a finite group scheme, then $A/K(L)$ makes sense and is isomorphic to $\mathrm{Pic}^0(A)$. If L is ample, we have seen this above (modulo the variety structure on $A/K(L)$).

2. If L is non-degenerate, then there exists a non-negative integer $i(L)$ called the index of L such that $H^{i(L)}(A, L) \neq 0$ and $H^j(A, L) = 0$ for $j \neq i(L)$. If L is ample, then (believing this result), clearly $i(L) = 0$.

3. Suppose L is non-degenerate. Let M be any ample line bundle on A. Define
$$p(n) = \chi(A, L \otimes M^n) \qquad n \in \mathbb{Z}.$$
We know from standard algebraic geometry that p is a polynomial in n, the so-called Hilbert polynomial of L w.r.t. M. Then $p \in \mathbb{Q}[t]$ has all its roots in $\mathbb{R}$ and the number positive roots is equal to $i(L)$.