Lecture 18

Let J^g be the scheme constructed by gluing the various W_L as we did before. We want to give a more complete proof that J^g represents the functor $\mathrm{Pic}^g_{X/k}$ defined by

$$T \longmapsto \frac{\mathrm{Pic}^g(X_T)}{\mathrm{Pic}(T)}$$

$\leftarrow$ family of line bundles of deg g on X parametrised by T.

1. Let W be as before, i.e. W is the open subscheme of $X^{(g)}$ consisting of points $w \in X^{(g)}$ such that

$$H^1(X_w, \mathcal{O}_{X_w}(D_w)) = 0$$

where D_w is the effective degree g divisor on X_w represented by w. In other words

$$D_w = \mathcal{D}_u \big|_{X_w}$$

$$\begin{tikzcd} \mathcal{D}_u \arrow[r, hook] \arrow[dr] & X \times X^{(g)} \arrow[d] \\ \mathrm{Spec}\, k(w) \arrow[r, "w"'] & X^{(g)} \end{tikzcd}$$

<u>Claim</u>: Let T be a k-scheme (not necessarily of finite type over k) and $\mathcal{L}$ a line bundle on X_T such that

(i) $\deg L_t = g \quad \forall\, t \in T$

(ii) $H^1(X_t, L_t) = 0 \quad \forall\, t \in T$

Here $L_t = \mathcal{L}|_{X_t}$. Then $\exists!$ map $r: T \longrightarrow W$ and a line bundle M on T such that

$$(1\times\gamma)^*\mathcal{O}(\mathcal{D}_u|_{X_W}) \simeq \mathcal{L}\otimes p_2^* M.$$

Proof of claim:

$$\begin{tikzcd} X_T \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ T \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

By semicontinuity

$$R^1 p_{2*}\mathcal{L} = 0.$$

Again by semi-continuity we therefore see that $p_{2*}\mathcal{L}$ is a line bundle $\mathcal{N}$ on T. Let $M = \mathcal{N}^{-1}$. Then $p_{2*}(\mathcal{L}\otimes p_2^* M) \cong \mathcal{O}_T$ (by the projection formula) and hence $\exists$ a nowhere vanishing section s of $p_{2*}(\mathcal{L}\otimes p_2^* M)$. Now

$$H^0(X_T, \mathcal{L}\otimes p_2^*M) = H^0(T, p_{2*}(\mathcal{L}\otimes p_2^*M))$$

and s can be regarded as a section of $\mathcal{L}\otimes p_2^*M$. When we regard s as a section of $\mathcal{L}\otimes p_2^*M$ we will use the symbol σ for it.

Since $R^1 p_{2*}(\mathcal{L}\otimes p_2^*M) = (R^1 p_{2*}\mathcal{L})\otimes M = 0$, therefore

$$p_2(\mathcal{L}\otimes p_2^*M)\otimes_{\mathcal{O}_T} k(t) = \Gamma(X_t, \mathcal{L}_t) \qquad \forall t\in T$$

Hence $s(t)$ can be identified with $\sigma|_{X_t}$. The fact that $s(t)$ is never zero (for any $t\in T$) amounts to saying $\sigma|_{X_t}$ is not the zero section of L_t. Write σ_t for $\sigma|_{X_t}$.

Let $\mathcal{D}\subset X_T$ be the effective Cartier divisor given by σ. Let $D_t = \mathcal{D}|_{X_t} = Z(\sigma_t)$, the zero locus of σ_t. Clearly D_t is an effective divisor on X_t of degree g, moreover, $\mathcal{D}\longrightarrow T$ is flat (local criterion for flatness) and hence we have a map (by the universal property of $X^{(g)}$) $\gamma : T \longrightarrow X^{(g)}$ s.t. $\mathcal{D} = (1\times\gamma)^{-1}(\mathcal{D}_u)$. Our hypothesis on $\mathcal{L}$ then forces γ to factor through W. (Note $L_t \cong \mathcal{O}(D_t)$). This proves the claim, for

$$(1\times\gamma)^* \mathcal{O}(\mathcal{D}_u|_{X_W}) \simeq \mathcal{O}(\mathcal{D}) \cong \mathcal{L}\otimes p_2^* M. \quad //$$

Let us now show that J^g represents $\mathrm{Pic}^g_{X/k}$.

Let T be a k-scheme (not necessarily of finite type) and $\mathcal{L}$ a line bundle on X_T s.t. L_t is of degree g on X_t for every $t \in T$.

Let $s \in T$. Since $h^0(L_s) > 0$ $\left(\begin{aligned} h^0(L_s) &= h^1(L_s) + g+1-g \\ &= h^1(L_s)+1 > 0 \end{aligned} \right)$ therefore we have at least one effective divisor D_s on X_s such that $\mathcal{O}(D_s) \cong L_s$. By the universal property of $X^{(g)}$ we have a map

$$s = \mathrm{Spec}(k(s)) \longrightarrow X^{(g)}$$

such that D_s is the pull-back of $\mathcal{D}_u$. Let w be the image of s in $X^{(g)}$. Let w^* be any closed point in the closure of w, and let L be a line bundle on X s.t. $L \simeq \mathcal{O}(D_{w^*})$. Clearly the pull-back of L to X_s is $\mathcal{O}(D_s)$.

For any line bdle M on X, define

$$T_M = \{ t \in T \mid H^1(X_t, L_t \otimes p_1^*(\mathcal{O}(D_0)\otimes M^{-1})) = 0 \}$$

It is clear that $s \in T_L$, where s and L are as above.

From our earlier calculations we have a map $T_L \longrightarrow W$ arising from $\mathcal{L} \otimes p_1^*(\mathcal{O}(D_0) \otimes M^{-1})|_{X_{T_L}}$. This really is a map $T_L \longrightarrow W_L$ and hence a map $T_L \xrightarrow{\gamma_L} U_L$ where U_L is the image of W_L in $J^{\mathcal{F}}$.

What we have shown is:

(a) $\{T_L\}$ is an open cover of T as L varies over line bundles in X

(b) The γ_L's glue to give $\gamma : T \longrightarrow J^{\mathcal{F}}$.

It is easy to see that γ is the required classifying map.