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Lecture 17

Last time we had glued together open subschemes W_L of $X^{(g)}$ (X a smooth complete curve of genus g over a field k). Recall that W_L is a copy $W = \{ w \in X^{(g)} \mid H^1(X_w, L_w) = 0 \}$, where L_w is the line bundle $\mathcal{O}(D_w)$. L varies over line bundles of degree g on X. If M is another line bundle of degree g on X, we have an open subscheme W_{LM} of W_L given by

$$W_{LM} = \{ w \in W_L \mid H^1(X_w, \mathcal{O}(D_w) \otimes M \otimes L^{-1}) = 0 \}.$$

It is not hard to see that the following diagram commutes for L, M, N line bundles on X of degree g

$$\begin{tikzcd}
W_{ML} \cap W_{MN} \arrow[rr, "\varphi_{LN}"] \arrow[dr, "\varphi_{LM}"'] & & W_{NL} \cap W_{NM} \\
& W_{LM} \cap W_{LN} \arrow[ur, "\varphi_{MN}"'] &
\end{tikzcd}$$

where $\varphi_{LM} : W_{ML} \longrightarrow W_{LM}$ is the map described last time. In naive terms it is described as follows: Let $w \in W_{ML}$ so that $h^1(\mathcal{O}(D_w) \otimes L \otimes M^{-1}) = 0$. Then $\exists !$ effective divisor w' of degree g on X such that $\mathcal{O}(D_{w'}) \cong \mathcal{O}(D_w) \otimes L \otimes M^{-1}$. It is clear that w' lies in W_{LM} for $H^1(\mathcal{O}(D_{w'}) \otimes M \otimes L^{-1}) = 0$.

Thus the W_L's glue to give a k-scheme J^g. We will assume in what follows that $k = \bar{k}$. We fix an effective degree g divisor D_0 such that $H^1(X, \mathcal{O}(D_0)) = 0$, and write $w_0 \in W$ for the corresponding point on W.

Let U_L be the image of W_L in J^g.

$$\coprod W_L \longrightarrow J^g$$

Note that since the arrow is obtained via a glueing process, it is an open map, and U_L is open in J^g.

On $X \times W_L$ set

$$\mathcal{L}_L = p_1^*(L \otimes \mathcal{O}(-D_0)) \otimes \mathcal{O}(\mathcal{D}_u)\Big|_{X \times W_L}$$

Here $\mathcal{D}_u$ is the universal divisor on $X \times X^{(g)}$, and we are regarding $W_L = W$ as an open subscheme of $X^{(g)}$. This descends to a line bundle on $X \times U_L$ since $U_L \simeq W_L$.

If $u_L \in U_L$ is the point corresponding to $w_0 \in W = W_L \xrightarrow{\sim} U_L$, then one checks that

$$\mathcal{L}_L\Big|_{X \times \{u_L\}} = L.$$

One can regard $\mathcal{L}_L$ as line bundle on $X \times U_L$. It is easy to see the $\mathcal{L}_L$ glue to give a line bundle on $X \times J^g$.

Claim: $(J^g, \mathcal{L})$ represents $\mathrm{Pic}^g_{X/k}$.

This means, given a k-scheme T and a line bundle $\mathcal{M}$ on $X_T = X \times T$ such that the restrictions of $\mathcal{M}$ to X_t is a line bundle of degree g for every $t \in T$, then $\exists!$ $T \xrightarrow{\gamma} J^g$ and a line bundle Q on T such that

$$(1 \times \gamma)^* \mathcal{L} \cong \mathcal{M} \otimes p_2^* Q.$$

Suppose the claim is true.

On $X^{(g)}$ we have $\mathcal{O}(D_u)$ and we know that for every $q \in X^{(g)}$, $\mathcal{O}(D_u)\big|_{X_q} = \mathcal{O}(D_q)$ is a line bundle of degree g. Therefore (if we admit the claim) we have a map $X^{(g)} \longrightarrow J^g$. This map is surjective. Indeed, if $s \in J^g$, then $\mathcal{L}\big|_{X_s}$ is a line bundle of degree g and if D is any effective degree g divisor on X_s such $\mathcal{O}(D) \simeq \mathcal{L}|_{X_s}$. If $q \in X^{(g)}$ is the point corresponding D, then clearly q maps to s.

This shows that J^g is of finite type, for it is locally of finite type (each U_L is of finite type) and being the image of a quasi-compact scheme, must be quasi-compact.

Let us now show that J^g is proper over k

(admitting the claim). Let R be a d.v.r over k and K its quotient field. Suppose we have a map $\operatorname{Spec} K \longrightarrow J^g$. This is equivalent to having a line bundle L_K on X_K ($X_K = X \times_k \operatorname{Spec} K$), of degree g. Let D_K be an effective divisor on X_K corresponding to L_K. Let D_R be the closure of D_K in X_R.

$$\begin{tikzcd}
D_K & \\
X_K \arrow[r, hook] \arrow[d] & X_R \arrow[d] \\
\operatorname{Spec} \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

D_R must be flat over $\operatorname{Spec} R$, since it is surjective and hence its structure sheaf is torsion free.

It follows that $D_R \longrightarrow \operatorname{Spec} R$ is finite and flat. Let $L_R = \mathcal{O}(D_R)$. This is a line bundle on X_R and the restriction to each fibre is a line bundle of degree g. In greater detail, since $D_R \longrightarrow \operatorname{Spec} R$ is a finite map, D_R is affine, say $D_R = \operatorname{Spec} A$. Then D_R being flat implies $R \longrightarrow A$ is flat, and "finite" implies A is f.g. as an R-module. R being local, therefore A is free. The rank of $A = \dim_{k(\mathfrak{p})} (A \otimes_R k(\mathfrak{p}))$ for every $\mathfrak{p} \in \operatorname{Spec} R$. Take $\mathfrak{p}$ to be the generic pt of R, and we get $\operatorname{rank}_R(A) = \dim_K (A \otimes_R K)$. However, $\operatorname{Spec}(A \otimes_R K) = D_K$, and D_K is divisor of deg g. Hence $\dim_K (A \otimes_R K) = g$. Thus $\operatorname{rank} A = g$.

Thus L_R is a family of deg g l.b's parametrised by

Spec R. By the univ prop. of J^g (using the claim) we get a unique map $\operatorname{Spec} R \longrightarrow J^g$ s.t. L_R is the pull back of $\mathcal{L}$.

Moreover, if $\mathcal{P} = \mathcal{L} \otimes p_1^* \mathcal{O}(-D_0)$, then $\mathcal{P}$ is a family of degree 0 line bundles on X parametrised by J^g. $(J^g, \mathcal{P})$ then represents $\mathrm{Pic}^0_{X/k}$.

In other words $\mathrm{Pic}^0_{X/k}$ is representable, and the representing object is called the Jacobian and denoted J_X or $J_{X/k}$.

It remains to prove the claim:

Let $T \in \mathrm{Sch}/k$ and $\mathcal{M}$ a family of line bundles on X of degree g parametrised by T (i.e. $\mathcal{M}$ is a l.b. on X_T and $\mathcal{M}|_{X_t}$ is of degree g $\forall\, t \in T$).

Let $M_t = \mathcal{M}|_{X_t}$. Let $s \in T(k)$. Let M_s. Consider open set on T given by

$$T_s = \{ t \in T \mid H^1(X_t, M_t \otimes M_s^{-1} \otimes \mathcal{O}(D_0)) = 0 \}.$$

We will find a map $T_s \longrightarrow U_{M_s}$ and these maps glue as s varies over $T(k)$. This will give us a map $T \longrightarrow J^g$.