March 9, 2021

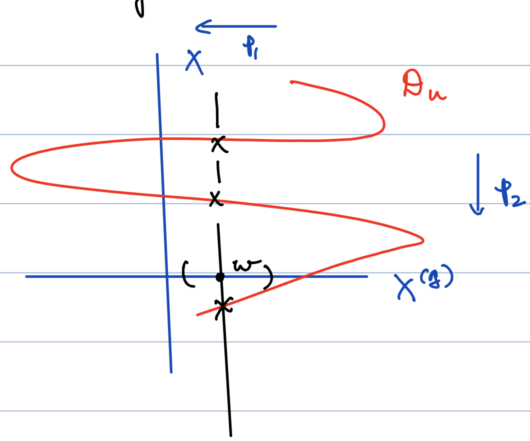
Lecture 16

Let k be a field, X a complete smooth curve of genus g with at least one k-rational point.

We have seen that if $k=\bar{k}$ then there exists at least one effective divisor of degree g such that $H^1(X,L)=0$ where L is the line bundle corresponding the effective divisor.

This means (again with $k=\bar{k}$), if U is the locus in $X^{(g)}$ on which $H^1(X,\mathcal{O}(D))$ vanishes $(D\in U)$, then U is non-empty. By semi-continuity U is open.

Now drop the hypothesis that $k=\bar{k}$. We have a universal degree g divisor $\mathcal{D}_u \subseteq X\times X^{(g)}$, and what the above shows us is that the open set where $R^1 p_{2*}\mathcal{O}(\mathcal{D}_u)$ vanishes is non-empty (since its base change under $k \longrightarrow \bar{k}$ is non-empty).



Let us call this locus W.

By semi-continuity, if $w\in W$, then

$$H^1(X_w, \mathcal{O}_{X_w}(D_w))=0$$

where the notation is self-explanatory.

We will use this later.

Let us return to the case where $k=\bar{k}$.

To fix notations $W\subseteq X^{(g)}$ is the open subscheme on which $h^1(\mathcal{O}_{X_w}(D_w))=0$.

Since $k = \bar{k}$ and since W is of finite type ($X^{(g)}$ is of finite type) there $W(k) \neq \phi$.

Fix a k-rational point in W, say w_0. Let $D_0 = D_{w_0}$ be the corresponding divisor on $X = X_{w_0}$.

Let $L_0 = \mathcal{O}(D_0)$. By R.R, since $H^1(X_w, \mathcal{O}(D_w)) = 0$ $\forall\, w \in W$, we have $h^0(\mathcal{O}(D_w)) = 1$ for all $w \in W$.

Indeed (with $L_w = \mathcal{O}(D_w)$)

$$h^0(L_w) - h^1(L_w) = \deg(L_w) + 1 - g$$

$$\Rightarrow \quad h^0(L_w) = g + 1 - g \qquad \left(\text{since } h^1(L_w) = 0, \ \& \ \deg L_w = g\right)$$

$$= 1.$$

This means that the only effective divisor in the complete linear system determined by L_w. Informally this means that W parametrises line bundles L of degree g with $h^1(L) = 0$. For if L is a l.b. of deg g with $h^1(L) = 0$, again by R.R. $h^0(L) = 1$.

Let $\widetilde{JJ} = \coprod_L W_L$

where each $W_L = W$, and L varies over isomorphism classes of line bundles of degree g.

Suppose L and M are two line bundles of deg g on X. Set W_{LM} to be the open locus of $W_L = W$ consisting of points $w \in W$ such that $\mathcal{O}(D_w) \otimes M \otimes L^{-1}$ has no H^1.

$$W_{LM} = \{ w \in W_L \mid H^1(X_w, \mathcal{O}(D_w) \otimes M \otimes L^{-1}) = 0 \}.$$

Note that $W_{LL} = W$.

Let $\varphi_{ML} : W_{ML} \longrightarrow W_{LM}$

be described as follows: Let $w \in W_{ML}$. Since $H^1(X_w, \mathcal{O}(D_w)\otimes L\otimes M^{-1}) = 0$ we have a unique point $w' \in W$ such that $\mathcal{O}(D_{w'}) = \mathcal{O}(D_w)\otimes L\otimes M^{-1}$.

Let $\varphi_{ML}(w) = w'$. Note that $\mathcal{O}(D_{w'})\otimes M\otimes L^{-1} \simeq \mathcal{O}(D_w)$. ,

Let $\varphi_{ML}(w) = w'$. Note that $\mathcal{O}(D_{w'})\otimes M\otimes L^{-1} \simeq \mathcal{O}(D_w)$, where $H^1(\mathcal{O}(D_{w'})\otimes M\otimes L^{-1}) = 0$, i.e, $w' \in U_{LM}$.

It remains to show that φ_{LM} is a map of varieties. This can be done by replacing w by a T-valued point

$$w : T \longrightarrow W_{ML}$$

and setting $w' : T \longrightarrow W_{LM}$ as indicated above, to get a map (functorial in T)

$$\varphi_{LM}(T) : W_{ML}(T) \longrightarrow W_{LM}(T)$$

By Yoneda get $W_{ML} \xrightarrow{\varphi_{ML}} W_{LM}$.

Note φ_{LM} is an isomorphism, the inverse being φ_{ML}

Suppose we have three line L, M, N of degree g. Check that

$$\begin{tikzcd}
W_{LM} \cap W_{LN} & & W_{NM} \cap W_{NL} \arrow[ll, "\varphi_{LN}"'] \arrow[dl, "\varphi_{MN}"] \\
& W_{ML} \cap W_{MN} \arrow[ul, "\varphi_{LM}"] &
\end{tikzcd}$$

Commutes. So the cocycle rules hold. It follows that

We have a scheme J^g by gluing the various W_L along the W_{LM}'s via the data $\{\varphi_{LM}\}$.

The problems that remain:

- Is J^g of finite type?
- Is it complete
- Is it projective?
- Does it represent $\mathrm{Pic}^g_{X/k}$?