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Lecture 15

Let X be a smooth complete curve over an algebraically closed field k. Let $X^{(n)}$ be the symmetric n-fold product of X with itself, i.e. $X^{(n)} = X^n/S_n$. The quotient means the following: let $U = \operatorname{Spec} A$ be an affine open subscheme of X, then $U^n/S_n := \operatorname{Spec}\left(\left(A^{\otimes n}\right)^{S_n}\right)$. These U^n/S_n glue as we vary U over affine open subschemes of X to give X^n/S_n. There is a natural map $X^n \xrightarrow{\pi} X^{(n)}$.

We saw that $X \times X^{(n)}$ has universal effective divisor of degree n, D_n.

$$\begin{tikzcd} D_n \arrow[r, hook] \arrow[dr] & X \times X^{(n)} \arrow[d, "p_2"] \\ & X^{(n)} \end{tikzcd}$$

finite and flat with fibres which are artin local rings of length n.

Conclusion: $X^{(n)}$ is the space of effective divisors of degree n on X. This is the so-called Hilbert scheme of effective divisors of degree n on X.

Remark: True even if k is not algebraically closed.

Last time we showed that the locus U on $X^{(g)}$ ($g = \text{genus}(X)$) on which $R^1 p_{2*} \mathcal{O}(D_n)$ vanishes is non-empty and open.

Indeed we know $\exists$ an effective divisor D of degree g s.t. $H^1(X, \mathcal{O}(D)) = 0$.

$$\text{Fact} \quad \{R^1 p_{2*} \mathcal{O}(\mathcal{D}_u)\}_t \otimes_{\mathcal{O}_{X^{(n)}}} k(t) = H^1(X \otimes k(t), \mathcal{O}(\mathcal{D}|_{X\otimes k(t)}))$$

This is just right exactness of $\otimes$-product since H^2's are zero on a curve.

Example: Let $g=1$. Let us drop the assumption that $k=\bar{k}$. However let us assume X has a k-rational point P_0. We will often regard P_0 as an effective divisor of degree 1.

If L is a line bundle of degree 1. By Serre duality $h^1(L) = h^0(\omega_X \otimes L^{-1})$. Since $\deg \omega_X = 2g-2 = 0$, therefore $\deg(\omega_X \otimes L^{-1}) = -1$ and hence $h^0(\omega_X \otimes L^{-1}) = 0$. In particular by R.R.

$$h^0(L) = \deg(L) + 1 - g = 1 + 1 - 1 = 1.$$

This means any two non-zero sections s & σ of L differ by scalar multiple from k^*, $s = \lambda \cdot \sigma$, $\lambda \in k^*$. It follows that the zero loci of s & σ coincide. Hence there is only effective divisor D representing L. Moreover, since $\deg D = \deg L = 1$, D can be regarded as k-rational point on X. This shows that given an effective divisor D on X of degree 1, if D' is another effective divisor linearly equivalent to D, then $D' = D$.

Now $X = X^{(1)}$. The universal divisor is the diagonal Δ in $X \times X$. We can prove the universal property for Δ directly in this case. Here is how we do it.

$$\begin{tikzcd}
\Delta \arrow[r, hook] \arrow[ddr, "\sim"] & X \times_k X \arrow[r, "p_1"] \arrow[dd, "p_2"] & X \arrow[dd] \\
& & \\
T \arrow[r, dashed] & X \arrow[r] & \mathrm{Spec}\, k
\end{tikzcd}$$

Now suppose we have a k-scheme T and a family of degree 1 divisors $\mathcal{D}$ on T. This means $\mathcal{D}$ is a closed subscheme of $X_T = X \times_k T$ and $\mathcal{D} \longrightarrow T$ is finite flat with fibres which are single points.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, phantom, "\subseteq"] \arrow[dr, "\sim"] & X_T = X \times T \arrow[d, "p_2"] \\
& T
\end{tikzcd}$$

Now, by semi-continuity $p_{2*}\mathcal{O}(\mathcal{D})$ is a line bundle. This is because $H^0(X_t, \mathcal{O}(\mathcal{D}_t))$ is one-dimensional for every t; and $H^1(X_t, \mathcal{O}(\mathcal{D}_t)) = 0$.

In greater detail, since the questions on T, assume $T = \mathrm{Spec}\, A$ and that we have a complex of f.g. proj modules

$$P^\bullet : \quad 0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow 0$$

such that $H^i(P^\bullet \otimes_A M) = H^i(X_T, \mathcal{O}(\mathcal{D}))$, $\forall i$.

By right exactness of $\otimes$-product, we get

$$H^1(P^\bullet \otimes_A M) = \mathrm{coker}\{P^0 \otimes_A M \longrightarrow P^1 \otimes_A M\}$$
$$= \mathrm{coker}\{P^0 \longrightarrow P^1\} \otimes_A M$$

Thus $H^1(X_T, \mathcal{O}(\mathcal{D}) \otimes M) \xrightarrow{\sim} H^1(X_T, \mathcal{O}(\mathcal{D})) \otimes M$.

In particular taking $M = k(t)$, $t \in T$, we get $H^1(X_T, \mathcal{O}(\mathcal{D}))\otimes_A k(t) = 0 \quad \forall t \in T$.

By Nakayama $H^1(X_T, \mathcal{O}(\mathcal{D})) = 0$.

Thus we have an exact sequence

$$0 \longrightarrow H^0(P^\cdot) \longrightarrow P^0 \longrightarrow P^1 \longrightarrow 0$$

This forces $H^0(P^\cdot)$ to be a projective module and also that for any A-module M,

$$0 \to H^0(P^\cdot)\otimes_A M \longrightarrow P^0 \otimes_A M \longrightarrow P^1 \otimes_A M \to 0 \quad \text{is exact}$$

This means

$$H^0(P^\cdot)\otimes_A M = H^0(P^\cdot \otimes_A M)$$

i.e.,

$$H^0(X_T, \mathcal{O}(\mathcal{D}))\otimes_A M = H^0(X_T, \mathcal{O}(\mathcal{D})\otimes_A M)$$

Taking $M = k(t)$, $t \in T$, we get

$$H^0(X_T, \mathcal{O}(\mathcal{D}))\otimes_A k(t) = H^0(X_t, \mathcal{O}(\mathcal{D}_t))$$

The right side is a $k(t)$-vector space of dimension 1, since genus $(X_t) = 1$, X_t has $k(t)$-rational pt & $\mathcal{O}(\mathcal{D}_t)$ is a line bdle of degree 1.

Hence the projective A-module $H^0(X_T, \mathcal{O}(\mathcal{D}))$ has rank 1.

Let $\mathcal{M} = p_{2*}\mathcal{O}(\mathcal{D})$.

Let $\mathcal{L} = \mathcal{O}(\mathcal{D}) \otimes p_2^*\mathcal{M}^{-1}$. $\longleftarrow$ The family of l.b's this represents is the same as the family given by $\mathcal{O}(\mathcal{D})$.

Then, by the projection formula

$$p_{2*}\mathcal{L} = p_{2*}\mathcal{O}(\mathcal{D}) \otimes \mathcal{M}^{-1} \simeq \mathcal{O}_T.$$

Hence we have a nowhere vanishing section s of $p_{2*}\mathcal{L}$.

So $0 \neq s \in H^0(T, p_{2*}\mathcal{L})$.

However,

$$H^0(T, p_{2*}\mathcal{L}) = H^0(X_T, \mathcal{L})$$

So s is also a section of $\mathcal{L}$. To avoid confusion we write $\sigma \in H^0(X_T, \mathcal{L})$ when we think of s as an element of $H^0(X_T, \mathcal{L})$.

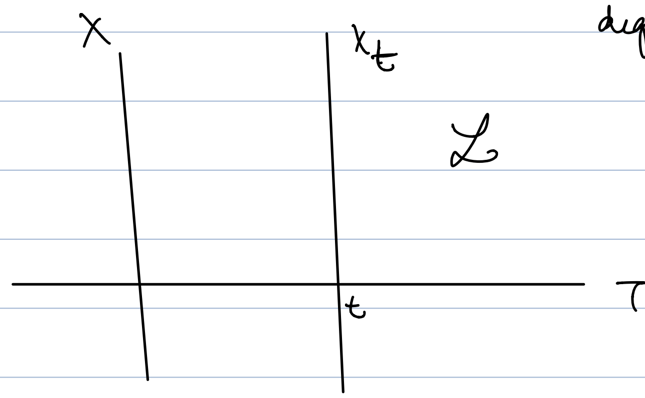
The above computations, while useful, are not what we need.

We know $\mathcal{D} \longrightarrow T$ is an isomorphism. But $\mathcal{D} \subseteq X \times T$. So we have the composite

$$\begin{tikzcd} T \arrow[r, "\sim"] \arrow[rrr, bend right, "\gamma"'] & \mathcal{D} \arrow[r, hook] & X \times T \arrow[r, "p_1"] & X. \end{tikzcd}$$

So we have a map γ. It is easy to see that $(1 \times \gamma)^{-1}(\Delta) = \mathcal{D}$.

Now suppose we have a line $\mathcal{L}$ on X_T where T is a k-scheme, of degree 1, i.e. $L_t := \mathcal{L}|_{X_t}$ is of degree 1.



Replace $\mathcal{O}(\mathcal{D})$ by $\mathcal{L}$ in the above computations and we see that the following is true:

① $R^1 p_{2*}\mathcal{L} = 0$

② $p_{2*}\mathcal{L}$ is a line bundle &

$$(p_{2*}\mathcal{L}) \otimes_{\mathcal{O}_T} \mathcal{F} \simeq p_{2*}(\mathcal{L} \otimes p_2^* \mathcal{F})$$

$\forall$ q-c sheaves $\mathcal{F}$ on T.

By modifying $\mathcal{L}$ if necessary (by using $\mathcal{M}$ as we did earlier), we may assume

$$p_{2*}\mathcal{L} \xrightarrow{\sim} \mathcal{O}_T.$$

So we have a nowhere vanishing section $s \in H^0(T, p_{2*}\mathcal{L}) = H^0(X_T, \mathcal{L})$, and let us write s as σ when we regard s as a section of $\mathcal{L}$.

Let $t \in T$. The fact that $s(t) \neq 0$ is interpreted as $\sigma|_{X_t}$ is non-zero, i.e., σ cannot vanish on the entire X_t. Write σ_t for $s(t)$ when we see this as a section of $L_t = \mathcal{L}|_{X_t}$. Since L_t has degree 1, σ_t vanishes at exact one point $D_t \in X_t$.

Let $\mathcal{D}$ be the divisor given by σ. Note $\mathcal{L} = \mathcal{O}(\mathcal{D})$. What said in the last paragraph amounts to showing $\mathcal{D} \longrightarrow T$ is flat (by the local criterion for flatness). Since the fibre is a single point, $\mathcal{D} \xrightarrow{\sim} T$ is an isomorphism (check!)

Hence we get a map $T \xrightarrow{r} X$ as the composite

$$T \xrightarrow{\sim} \mathcal{D} \xrightarrow{2^{nd} \text{ projection}} X.$$

Easy to see again that $\mathcal{D} = (1\times\gamma)^{-1}(\Delta)$, & hence $\mathcal{L} = (1\times\gamma)^* \mathcal{O}(\Delta)$.

Now suppose $\mathcal{L}$ is a family of degree 0 line bundles on X parametrised by T.

$$\begin{tikzcd} \mathcal{L} \arrow[r] & X_T \arrow[r] \arrow[d] & X \arrow[d] \\ & T \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Fix a k-rational point D_0 on X (as we did earlier).

Consider $\mathcal{L}' = \mathcal{L} \otimes p_1^* \mathcal{O}(D_0)$.

Then $\mathcal{L}'$ is a family of degree 1 line bundles on X. Hence we get a ! map $\gamma : T \longrightarrow X$ such that

$$(1\times\gamma)^* (\mathcal{O}(\Delta)) \otimes p_1^* (\mathcal{O}(D_0))^{-1} \simeq \mathcal{L} \otimes p_2^* M$$

for some line bundle M.

This essentially proves that X is its own Jacobian.

It represents $\operatorname{Pic}^0_{X/k}$

Hence $\operatorname{Pic}_{X/k}(T) = \dfrac{\operatorname{Pic}(X_T)}{\operatorname{Pic}(T)}$

$\operatorname{Pic}^0_{X/k}$ is the subfunctor of $\operatorname{Pic}_{X/k}$ for $\deg 0$ line bundles