March 2, 2021

Lecture 14

Last time we showed that if X is a smooth complete curve over a field k, say $k=\bar{k}$, then given a line bundle L on X with $H^0(L)\neq 0$, then $\exists$ a closed point $P\in X$ such that $\dim_k H^0(L\otimes \mathcal{O}(-P)) = \dim_k H^0(L)-1$.

Take $L=\omega_X$. If the genus g of $X\geq 1$, then $H^0(L)\neq 0$. In fact $h^0(L)=h^0(\omega_X)=g\geq 1$. Therefore, by applying the result in the first paragraph iteratively, we see that we can find $P_1,\dots,P_g\in X(k)$, not necessarily distinct, such that if $D=P_1+P_2+\dots+P_g$ then $H^0(\omega_X\otimes\mathcal{O}(-D))=0$. By Serre duality, this means

$$H^1(X,\mathcal{O}(D))=0.$$

The space of effective divisors of X:

For $n\geq 1$, let

$$X^n = \underbrace{X\times\dots\times X}_{n}$$

and let

$$X^{(n)} = X^n/S_n$$

If $U=\operatorname{Spec} A$ is an affine open subscheme of X, then $U^n=\underbrace{U\times\dots\times U}_{n}$ is the spectrum of $\underbrace{A\otimes_k\dots\otimes_k A}_{n}$.

Set

$$U^{(n)} = \operatorname{Spec}\{(A^{\otimes n})^{S_n}\}$$

where the action of S_n on $A^{\otimes n}$ is the natural one, namely, if $\sigma \in S_n$, $\sigma(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$.

It is not hard to show that the $U^{(n)}$ patch as U varies over affine open subschemes of X. The resulting glued scheme is $X^{(n)}$.

Note we have $U^n \longrightarrow U^{(n)}$ is surjective since $\{A^{\otimes n}\}^{S_n} \hookrightarrow A^{\otimes n}$. These maps patch to give a surjective map

$$X^n \longrightarrow X^{(n)}. \qquad \text{———} (*)$$

It is not hard, from basic commutative algebra, that $U^{(n)}$ is a finite type integral scheme over k. The surjective map $(*)$ then shows that $X^{(n)}$ is proper over k. Since $(*)$ is finite (the generic fibre cardinality is n), $X^{(n)}$ is projective.

Finally, using symmetric polynomials, one can show $U^{(n)}$ is regular, i.e smooth over k, and hence $X^{(n)}$ is smooth over k.

Example: Suppose $U = \mathbb{A}^1 = \operatorname{Spec} k[T]$. Then $U^n = \operatorname{Spec} k[T_1, \ldots, T_n]$. Check that $k[T_1, \ldots, T_n]^{S_n}$ is $k[\sigma_1, \ldots, \sigma_n]$, where the σ_i are the elementary symmetric polynomials. It follows that $k[T_1, \ldots, T_n]^{S_n}$ is smooth over k. More generally, one can find a open cover $\mathcal{U}$ of X such that $U \in \mathcal{U} \Rightarrow U$ is affine & there is an étale map $U \longrightarrow \mathbb{A}^1$. The problem can be transferred to $\mathbb{A}^1$ where we have already solved it.

The net result is that we $X^{(n)}$ is a smooth projective variety over k; we have finite surjective map

$$X^n \xrightarrow{\pi} X^{(n)}.$$

A closed point D of $X^{(n)}$ is the image of a closed point $(P_1,\dots,P_n) \in X^n$. The closed point D of $X^{(n)}$ is identified with the divisor $P_1+\dots+P_n$ on X.

$X^{(n)}$ is identified with the space of effective divisors on X of degree n.

Remarks:

1. Suppose we have map of k-schemes

$$f : X^n \longrightarrow Z$$

such that $f = f\circ\sigma$, $\forall\, \sigma \in S_n$. {More precisely, if $T \xrightarrow{\xi} X^n$, i.e. $\xi \in h_{X^n}(T) = h_X(T)\times\dots\times h_X(T)$ then for every $\sigma \in S_n$, we have $\sigma\xi \in h_{X^n}(T)$, since S_n acts on $h_X(T)\times\dots\times h_X(T)$. This is functorial in T, whence we have an action of S_n on X^n (by Yoneda). Therefore it makes sense to talk about $f\circ\sigma$. The requirement is that $f = f\circ\sigma$.} Then $\exists!\ \bar{f} : X^{(n)} \longrightarrow Z$ s.t.

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$$\begin{tikzcd}
X^n \arrow[d, "\pi"'] \arrow[dr, "f"] & \\
X^{(n)} \arrow[r, "\bar{f}"'] & Z
\end{tikzcd}.$$

2. There is universal effective divisor on $X \times X^{(n)}$. In greater detail, there exists an effective divisor $\mathcal{D}$ on $X \times X^{(n)}$, such that $\mathcal{D}_u \xrightarrow{p_2} X^{(n)}$ (via projection to the 2nd factor) is finite, flat, and $p_{2*}(\mathcal{O}_{\mathcal{D}})$ is locally free of rank n, and if $\theta \in X^{(n)}$, then $\mathcal{D}_u|_{X \times \{\theta\}}$ is an effective divisor of degree n on X. Moreover, if $T \in Sch_k$, and $\mathcal{D}$ an effective (Cartier) divisor on $X \times_k T$, such that $\mathcal{D}$ is flat over T, and the fibres over T are degree n effective divisors on X, then $\exists!$ map $\gamma : T \longrightarrow X^{(n)}$ such that $(1 \times \gamma)^{-1}(\mathcal{D}_u) = \mathcal{D}$.

$$
\begin{tikzcd}
\mathcal{D}_u \arrow[r, hook] \arrow[dr, "\substack{\text{flat} \\ \text{finite}}"'] & X \times X^{(n)} \arrow[d] & X = X \times \{\theta\} \arrow[l, hook'] \arrow[d] & X \times T \arrow[ll, bend right] \arrow[d] & \mathcal{D} \arrow[l, hook'] \arrow[dl, "\substack{\text{finite flat} \\ \text{of degree } n}"] \\
 & X^{(n)} & \{\theta\} \arrow[l, hook'] & T \arrow[ll, bend left, "\gamma"] &
\end{tikzcd}
$$

In particular, if $D = P_1 + \cdots + P_n$ is a degree n effective divisor on X, then taking $T = \operatorname{Spec} k$, & $\mathcal{D} = D$. By the universal property $\exists!$ $\gamma : \operatorname{Spec} k \longrightarrow X^{(n)}$ such that $(1 \times \gamma)^{-1}(\mathcal{D}_u) = D$.

What is $\mathcal{D}_u$? In naive terms, $\mathcal{D}_u$ consists of points $(x, D) \in X \times X^{(n)}$ such that $x \in D$. This can be made sense of in terms of functor of points. The universal property is not so easy.

Theorem: Let $p_2 : X \times X^{(g)} \longrightarrow X^{(g)}$ be the canonical projection and let $L_u = \mathcal{O}(-\mathcal{D}_u)$. Then the open subscheme $\mathcal{U}$ of $X^{(m)}$ on which $R^1 p_{2*} L_u$ vanishes is non-empty.

Remark: The locus on which $R^1 p_{2*} L_u$ vanishes is open because of the semi-continuity theorem. Since $H^2(X \times \{\theta\}, L_u|_{X \times \{\theta\}}) = 0 \;\forall\, \theta \in X^{(n)}$, by semi-continuity (a) $R^2 p_{2*} L_u = 0$ and (b) $H^1(X \times \{\theta\}, L_u|_{X \times \{\theta\}}) = R^1 p_{2*} L_u \otimes_{\mathcal{O}_{X^{(n)}}} k(\theta)$.

Proof: We have seen that $\exists$ an effective divisor D of deg g s.t. $H^1(X, \mathcal{O}(D)) = 0$, and hence $D \in \mathcal{U}$.

Added after the lecture:

Recall $h_{X^m}(T) = \underbrace{h_X(T) \times \dots \times h_X(T)}_{m}$.

Define $\mathcal{D}'(T) \subseteq h_{X^{n+1}}(T)$ by

$$\mathcal{D}'(T) = \left\{ (x_0, x_1, \dots, x_n) \in h_{X^{n+1}}(T) \;\middle|\; x_0 = x_i \text{ for some } i > 0 \right\}$$

In other words, if $\pi_0, \pi_1, \dots, \pi_n$ are the projections from $X^{n+1} \longrightarrow X$, and Z_i is the closed subscheme of X^{n+1} given by $Z_i = \{ \xi \in X^{n+1} \mid \pi_0(\xi) = \pi_i(\xi) \}$ $i = 1, \dots, n$, then D' is represented by $\bigcup_{i=1}^{n} Z_i$. Note that locally on $U^n = \operatorname{Spec}(A^{\otimes n})$, the ideal sheaf of Z_i in U^n is generated by

$$s \otimes 1 \otimes \cdots \otimes 1 - \underset{i^{th}\text{ spot}}{1 \otimes \cdots \otimes s} \otimes 1 \cdots \otimes 1.$$

and that as U varies over an affine open cover of X, the sets U^n cover D'. D' is stable under the action of S_n.

D_n is the image D' under the map

$$X \times_k X^n \xrightarrow[1\times\pi]{} X \times_k X^{(n)}.$$