Feb 18, 2021

Lecture 13

Suppose $\mathscr{C}$ is category and X an object in $\mathscr{C}$. A composition law on X is a functorial map

$$\gamma : h_X \times h_X \longrightarrow h_X .$$

Functoriality means the following: For each $T \in \mathscr{C}$ we have

$$\gamma_T : h_X(T) \times h_X(T) \longrightarrow h_X(T)$$

such that if $T' \longrightarrow T$ is a morphism in $\mathscr{C}$, TFDC

$$\begin{tikzcd} h_X(T) \times h_X(T) \arrow[r, "\gamma_T"] \arrow[d] & h_X(T) \arrow[d] \\ h_X(T') \times h_X(T') \arrow[r, "\gamma_{T'}"'] & h_X(T') \end{tikzcd}$$

where the downward arrows are induced by $T' \longrightarrow T$.

If the composition law on X is such that $h_X(T)$ is group for every $T \in \mathscr{C}$, then X is said to be a group object in $\mathscr{C}$. More precisely (X, γ) is group object in $\mathscr{C}$ in this case. Note that this is equivalent to saying h_X is a functor taking values in (Groups).

If $\mathscr{C}$ has products and a final object S, then for (X, γ) to be a group object in $\mathscr{C}$ is equivalent to having a map

$$m : X \times X \longrightarrow X$$

such that :

P.T.O $\longrightarrow$

1. Associativity: The diagram

$$\begin{tikzcd}
X \times X \times X \arrow[r, "(m, \mathbb{1})"] \arrow[d, "(\mathbb{1}, m)"'] & X \times X \arrow[d, "m"] \\
X \times X \arrow[r, "m"'] & X
\end{tikzcd}$$

commutes

(2) <u>Existence of a left identity</u>: There exists a section $e: S \longrightarrow X$ of the unique map $p: X \longrightarrow S$ such that TFDC

$$\begin{tikzcd}
X \arrow[r, "(p, \mathbb{1}_X)"] \arrow[drr, "\mathbb{1}_X"'] & S \times X \arrow[r, "e \times \mathbb{1}_X"] & X \times X \arrow[d, "m"] \\
& & X
\end{tikzcd}$$

(d) <u>Existence of a left inverse</u>: There exists a morphism $i: X \longrightarrow X$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "(i, \mathbb{1}_X)"] \arrow[d, "p"'] & X \times X \arrow[d, "m"] \\
S \arrow[r] & X
\end{tikzcd}$$

commutes.

If S is a scheme and $\mathcal{C} = Sch_{/S}$ then a group object in $\mathcal{C}$ is called called a <u>group scheme over S</u>.

If k is an algebraically closed field then an abelian variety is a group scheme over Spec k which is a variety (i.e. it is integral, complete and of finite type).

Theorem: Let k be a field and X a group scheme over k which is of finite type and complete. Then following are equivalent

(a) X is smooth and irreducible

(b) X is smooth and integral

(c) X is irreducible and geometrically reduced

(d) X is integral and geometrically reduced.

(e) $X \times_k \operatorname{Spec} L$ is an abelian variety over L for every algebraically closed field extn L of k

(f) $X \times_k \bar{k}$ is abelian variety over $\bar{k}$, where $\bar{k}$ is the alg. closure of k.

Proof:

(a) $\Rightarrow$ (b) If X is smooth it is reduced, and being an integral scheme is the same as being reduced & irreducible.

(b) $\Rightarrow$ (c). If L is an extension of k, then $X_L := X \otimes_k L$ is smooth over L and hence reduced. So X is geometrically reduced.

(c) $\Rightarrow$ (d): Obvious

(d) $\Rightarrow$ (c) Since X geometrically reduced, X_L is geometrically reduced. We have to show X_L is irreducible.

closure of k. Let $G = \operatorname{Gal}(\bar{k}/k)$, and $X_L = X \otimes_k L$.

revisit

G acts on X_L. The point $\varepsilon \in X(k)$ is k-rational, and hence it has only element in the conjugacy it represents in X_L, namely the identity $\varepsilon_L \in X(L)$.

Since X_L is smooth over L, its irreducible components are disjoint, say $X_L = X_0 \sqcup X_1 \sqcup \dots \sqcup X_r$, where X_0 is the connected component containing ε_L. Each X_i is a smooth variety.

Since G acts continuously on X_L, it permutes the connected components. However $G(\varepsilon_L) = \varepsilon_L$, therefore X_0 is stable under G. It follows easily that X_0/G (i.e. the image of X_0 in X under the base change map $X_L \longrightarrow X$) is disjoint from $\left(\bigcup_{i=1}^{r} X_i\right)/G$ (where the quotient is defined to be the image). Since X is connected, it follows that $X_L = X_0$. Thus X_L is a variety. In particular X_L is irreducible and hence X is geometrically irreducible.

(e) $\Rightarrow$ (f) is clear

(f) $\Rightarrow$ (a) Well known.

Definition: Let k be a field and X a finite type complete group scheme over k. X is called an abelian variety over k if it satisfies any of the equivalent conditions in the theorem.

Line bundles on a curve: Let X be a smooth complete curve over an alg. closed field k. Let L be a line bundle on X such that $H^0(L) \neq 0$. ~~Let $0 \neq s \in H^0(X, L)$.~~

~~Let D be the effective divisor given by the locus on which s vanishes, and~~ Let p be a point on X.

Consider the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-p) & \longrightarrow & \mathcal{O}_X & \longrightarrow & k(p) & \longrightarrow & 0 \\ & & \| & & & & \uparrow & & \\ & & m_p & & & & \text{sky-scraper sheaf at } p. & & \end{array}$$

Tensor this with L. Get

$$0 \longrightarrow L(-p) \longrightarrow L \longrightarrow k(p) \longrightarrow 0.$$

Now

$$\chi(L(-p)) + \chi(k(p)) = \chi(L)$$

So $\quad \chi(L(-p)) + 1 = \chi(L)$

Hence

$$h^0(L(-p)) - h^1(L(-p)) + 1 = h^0(L) - h^1(L).$$

So

$$\begin{aligned} h^0(L) - h^0(L(-p)) + 1 &= h^1(L) - h^1(L(-p)) \\ &= h^0(\omega_X \otimes L^{-1}) - h^0(\omega_X \otimes L^{-1} \otimes \mathcal{O}(p)) \\ &\quad - h^0(\omega_X \otimes L^{-1}) \end{aligned}$$

Since $L(-p)$ is a subsheaf of L,

$$m := h^0(L) - h^0(L(-p)) \geq 0.$$

Since $\omega_X \otimes L^{-1}$ is a subsheaf of $\omega_X \otimes L^{-1} \otimes \mathcal{O}(p)$

Therefore

$$n := h^0(\omega_X \otimes L^{-1} \otimes \mathcal{O}(p)) - h^0(\omega_X \otimes L^{-1}) \geq 0$$

Hence

$m+n = 1$.

So either $m=1$ and $n=0$ or $n=1$ and $m=0$.

Now pick $s \in H^0(L)$, with $s \neq 0$, and $p \in X$ such that $s(p) \neq 0$. Then it is clear that $s \in H^0(L)$ but $s \notin H^0(L(-p))$. It follows $m := h^0(L) - h^0(L(-p)) > 0$. Hence $m=1$ and $n=0$.

In conclusion:

Theorem: Let k be an alg. closed field, X a smooth complete variety over k, L a line bundle on X with $H^0(L) \neq 0$, then there exists a point $p \in X$ such that $h^0(L(-p)) = h^0(L) - 1$.

General remarks: Know $h^0(\omega_X) = g > 0$. So we can iteratively find points $P_1, P_2, \ldots, P_g$ such that if $D = P_1 + \cdots + P_g$, then $h^0(\omega_X \otimes \mathcal{O}(-D)) = 0$. It follows that $h^1(\mathcal{O}(D)) = 0$.

Conclusion: $\exists$ an effective divisor D of degree g such that $h^1(\mathcal{O}(D)) = 0$.

Example: Suppose $K \longrightarrow L$ is purely inseparable finite extension with $L \neq K$. Then $\operatorname{Spec} L$ is a K-variety and it is regular, since L is a regular local ring. Now consider the base-change diagram

$$\left.\begin{array}{ccc} \mathrm{Spec}(L\otimes_k L) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(k) \end{array}\right\}$$

Now L is purely insep over k, and hence $L\otimes_k L$ is of the form $L[T]/(T^{p^e})$, which is not reduced, and hence not a regular local ring.