Feb 16, 2021

Lecture 12

Let $\mathscr{C}$ be a category. For $X \in \mathscr{C}$ write h_X for the functor $\mathrm{Hom}_{\mathscr{C}}(-, X)$. The functor h_X is called the "functor of points".

Theorem (Yoneda): Let $F : \mathscr{C} \longrightarrow (\mathrm{Sets})$ be a contravariant functor. Then for each $X \in \mathscr{C}$, there is a bijective map

$$\mathrm{Hom}_{\hat{\mathscr{C}}}(h_X, F) \xrightarrow{\sim} F(X)$$

where

$\hat{\mathscr{C}}$ = The category of (Sets)-valued contravariant functors on $\mathscr{C}$. The objects of $\hat{\mathscr{C}}$ are contravariant functors $G : \mathscr{C} \longrightarrow \mathrm{Sets}$ and morphisms are natural transformations $G \xrightarrow{\theta} G'$, i.e., for each $T \in \mathscr{C}$, we have a map $G(T) \xrightarrow{\theta(T)} G'(T)$ s.t. whenever $S \longrightarrow T$ is a morphism in $\mathscr{C}$, TFDC

$$\begin{tikzcd}
G(T) \arrow[r, "\theta(T)"] \arrow[d] & G'(T) \arrow[d] \\
G(S) \arrow[r, "\theta(S)"'] & G'(S)
\end{tikzcd}$$

where the vertical arrows are induced by the morphism $S \longrightarrow T$.

Moreover this bijective map is functorial in X, i.e $\mathrm{Hom}_{\hat{\mathcal{C}}}(h_{\_}, F) \longrightarrow F$ is an equivalence of functors.

Proof:

Suppose $\xi \in F(X)$. Let $T \xrightarrow{f} X$ be a map. Then we have a map $F(X) \xrightarrow{F(f)} F(T)$. Let $\theta(T)(f) = F(f)(\xi)$. This gives us a map

$$\begin{aligned} h_X(T) &\longrightarrow F(T) \\ f &\longmapsto F(f)(\xi). \end{aligned}$$

One checks easily that this is a functorial map (as T varies). So we get a map of functors

$$\theta_\xi : h_X \longrightarrow F.$$

Conversely, given a functorial map

$$\theta : h_X \longrightarrow F,$$

set ξ_θ equal to the image of $1_X \in h_X(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ under $\theta(X)$. Then $\xi_\theta \in F(X)$.

One checks easily that $\xi \longmapsto \theta_\xi$ and $\theta \longmapsto \xi_\theta$ are "inverse" operations. This gives the main part of the theorem. The rest is easy. //

We have a functor

$$h : \mathcal{C} \longrightarrow \hat{\mathcal{C}}$$

where $\hat{\mathcal{C}}$ has been defined earlier (the category of contravariant (Sets)-valued functors on $\mathcal{C}$), namely

$$X \longmapsto h_X.$$

This map is an embedding of the category $\mathscr{C}$ into $\hat{\mathscr{C}}$. Indeed, if $h_T \longrightarrow h_X$ is a map in $\hat{\mathscr{C}}$, then applying this to T, we get $h_T(T) \longrightarrow h_X(T)$, and the image of $1_T \in h_T(T)$ in $h_X(T) = \mathrm{Hom}_{\mathscr{C}}(T, X)$ give us a map $T \longrightarrow X$. In particular if $h_X = h_T$, then one sees that $X = T$.

Definition: $F \in \hat{\mathscr{C}}$ is said to be representable if it is in the essential image of $h : \mathscr{C} \hookrightarrow \hat{\mathscr{C}}$.

In naive terms, F is representable if $F \simeq h_X$ for some X.

Suppose F is representable, say $h_X \xrightarrow[\theta]{\sim} F$ for some X and θ, with $X \in \mathscr{C}$. Let $\xi =$ the image of 1_X under $\theta(X)$. Note $\xi \in F(X)$. The pair (X, ξ) is said to represent F. Loosely, "X represents F".

A law of composition on $X \in \mathscr{C}$ is a functorial map

$$\gamma : h_X \times h_X \longrightarrow h_X$$

This means it is a collection of maps:

$$\gamma_T : h_X(T) \times h_X(T) \longrightarrow h_X(T)$$

one for each $T \in \mathscr{C}$. Functoriality means that for each morphism $T' \longrightarrow T$ in $\mathscr{C}$, TFDC (the downward arrows arising from $T' \to T$).

$$\begin{tikzcd} h_X(T) \times h_X(T) \arrow[r, "\gamma_T"] \arrow[d] & h_X(T) \arrow[d] \\ h_X(T') \times h_X(T') \arrow[r, "\gamma_{T'}"'] & h_X(T') \end{tikzcd}$$

If the law of composition on X is such that $h_X(T)$ is a group under γ_T for every T, then X is called a group object in $\mathcal{C}$. Equivalently X is a group object in $\mathcal{C}$ if h_X is a group functor, i.e. h_X takes values in the category of groups.

Now assume $\mathcal{C}$ has products and a final object S. This means, if $X, Y \in \mathcal{C}$, then $X \times Y$ is meaningful, and for $X \in \mathcal{C}$, there exists a unique map $X \longrightarrow S$. The notion of a product is as follows. If $X, Y \in \mathcal{C}$, then there exists an object $X \times Y$ in $\mathcal{C}$, together with maps $X \times Y \xrightarrow{p} X$, $X \times Y \xrightarrow{q} Y$ such that if $T \in \mathcal{C}$ and we have two maps $f: T \longrightarrow X$, $g: T \longrightarrow Y$, then $\exists!$ map $(f, g): T \longrightarrow X \times Y$ such that $p \circ (f, g) = f$, $q \circ (f, g) = g$. Technically, the product is $(X \times Y, p, q)$. It is unique up to unique isomorphism (if it exists). In this case it is not hard to see that $h_{X \times Y}$ is canonically isomorphic to $h_X \times h_Y$.

Now suppose $\mathcal{C}$ is as above and X is a group object on $\mathcal{C}$. Then $\gamma: h_X \times h_X \longrightarrow h_X$ translates to a map

$$m: X \times X \longrightarrow X.$$

Moreover, if for each $T \in \mathcal{C}$, $p_T: T \longrightarrow S$ is the unique map to the final object S, then we have maps

$$\varepsilon(T): h_S(T) = \{p_T\} \longrightarrow h_X(T)$$

given by $p_T \longmapsto e_T$, when e_T is the identity on T $h_X(T)$.

One sees then that the following holds:

(1) Associativity: TFDC.

$$\begin{tikzcd}
X \times X \times X \arrow[r, "(1, m)"] \arrow[d, "(m, 1)"'] & X \times X \arrow[d] \\
X \times X \arrow[r, "m"'] & X
\end{tikzcd}$$

commutes.

(2) The existence of a left identity: The following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "(p, 1_X)"] \arrow[drr, "1_X"'] & S \times X \arrow[r, "(\varepsilon, 1)"] & X \times X \arrow[d, "m"] \\
& & X
\end{tikzcd}$$

(3) The existence of a left inverse: There exists $i: X \longrightarrow X$ such that TFDC.

$$\begin{tikzcd}
X \arrow[r, "(i, 1)"] \arrow[d] & X \times X \arrow[d, "m"] \\
S \arrow[r, "\varepsilon"'] & X
\end{tikzcd}$$

One checks that left identities are right identities and left inverses are right inverses.

Let S be a scheme and let $\mathcal{C} = \mathrm{Sch}_{/S}$, the category of S-schemes. A group scheme $/S$ is a group object in $\mathrm{Sch}_{/S}$.