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Lecture 11

Theorem: Let A be an abelian variety over an alg. closed field k and L a homogeneous line bundle (i.e. $t_x^* L \simeq L$ $\forall\, x \in A(k)$) on A. If L is non-trivial then $H^i(A,L) = 0$ for all i.

Proof:

We already know that $H^0(A,L) = 0$.

Let, as usual $m : A \times A \longrightarrow A$ be the group operation map (addition). Let $s : A \longrightarrow A \times A$ be the map $a \longmapsto (a, 0)$. Then we a CD

$$\begin{tikzcd} A \arrow[r, hook, "s"] \arrow[rr, bend right, "1_A"'] & A \times A \arrow[r, "m"] & A \end{tikzcd}$$

Suppose $k > 0$ is an integer such that $H^i(A,L) = 0$ for all $i < k$. Recall, since $L \in \mathcal{J}(A)$ therefore

$$m^* L = p_1^* L \otimes p_2^* L.$$

By Kunneth and the above

$$H^k(A \times A, m^* L) \xrightarrow{\sim} \bigoplus_{i+j=k} H^i(A, L) \otimes H^j(A, L).$$

If $i + j = k$, then either i or j is strictly less than k (we are only worrying about i, j, k non-negative). It follows, from our choice of k that $H^k(A\times A, m^*L) = 0$. Now look at the diagram we had, namely:

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[rr, bend right, "1_A"'] & A\times A \arrow[r, "m"] & A \end{tikzcd}$$

This means that the identity map on $H^k(A,L)$ factors as $H^k(A,L) \xrightarrow{m^*} H^k(A\times A, m^*L) \xrightarrow{s^*} H^k(A,L)$. It follows that the identity map on $H^k(A,L)$ is zero, i.e. $H^k(A,L) = 0$. //

Curves:

Let C be a smooth complete curve over k (as usual $k=\bar{k}$). Let ω_C be the canonical bundle on C. Then we know the following

1. $H^i(C,\mathcal{F}) = 0$ for $i \geq 2$, $\mathcal{F}$ any sheaf.
2. $H^i(C,L) = H^{1-i}(C, \omega_C \otimes L^{-1})^*$ for $i=0,1$, and L a line bundle (Serre duality).
3. For any line bundle L
$$h^0(L) - h^1(L) = \deg(L) + 1 - g \qquad \text{(Riemann-Roch).}$$
Here $h^i(\mathcal{F}) = \dim_k(H^i(C,\mathcal{F}))$, $\mathcal{F} \in C_c$.

Note that if $H^0(C,L) \neq 0$, then $\deg(L) \geq 0$. This is so for the following reason: Let $0 \neq s \in H^0(L)$.

Then $D = Z(s)$ (the zero-locus of s) is an effective divisor such that $L \simeq \mathcal{O}(D)$. Now $\deg D \geq 0$ since D is effective. Since $\deg L = \deg D$, it follows that $\deg L \geq 0$.

Since

$$h^1(L) = h^0(\omega_C \otimes L^{-1}) \qquad \text{(Serre duality)}$$

therefore

$$\deg(\omega_C \otimes L^{-1}) < 0 \implies h^1(L) = 0.$$

i.e. $\deg(\omega_C) - \deg(L) < 0 \implies h^1(L) = 0$

i.e. $\boxed{\deg L > 2g-2 \implies h^1(L) = 0.}$ —— 1

Next suppose $\deg L > 2g-1$. Now,

$$\begin{aligned} h^0(L) - h^1(L) &= \deg L + (1-g) \\ &> 2g-1 + (1-g) \\ &= g \end{aligned}$$

Also $h^1(L) = 0$ from 1. Hence

$$h^0(L) > g \geq 0.$$

Hence $h^0(L) \neq 0$. So $L \simeq \mathcal{O}(D)$ for some effective divisor D. Let P be a point on C. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(D-P) \longrightarrow \mathcal{O}(D) \longrightarrow k(P) \longrightarrow 0$$

obtained by $\otimes$-ing the exact seq

$$0 \longrightarrow \mathcal{O}(-P) \longrightarrow \mathcal{O}_C \longrightarrow k(P) \longrightarrow 0$$

by $\mathcal{O}(D)$. Since $\deg(D-P) = \deg D - 1 > 2g-2$ (for $\deg D > 2g-1$). Hence $h^1(\mathcal{O}(D-P)) = 0$. So the exact sequence $0 \to \mathcal{O}(D-P) \longrightarrow \mathcal{O}(D) \longrightarrow k(P) \longrightarrow 0$ gives us an exact seq

$$0 \longrightarrow H^0(C, \mathcal{O}(D-P)) \longrightarrow H^0(C, L) \longrightarrow k \longrightarrow 0$$

(since $h^1(\mathcal{O}(D-P)) = 0$)

In particular the map

$$H^0(C, L) \otimes_k k(P) \longrightarrow L \otimes k(P) = k(P)$$

is surjective. In other words L is generated by global sections (for by Nakayama, this means that the natural map $H^0(C, L) \otimes_k \mathcal{O}_C \longrightarrow L$ is surjective).

Hence

$$\boxed{\deg L > 2g-1 \implies L \text{ is generated by global sections}} \text{ — } (2)$$

Now suppose $\deg L > 2g$. Let P, Q be two points on C. Suppose $P \neq Q$. Consider the exact sequence

$$0 \longrightarrow \underset{\substack{\| \\ (\mathcal{O}(D-P-Q) \\ \text{where } L = \mathcal{O}(D))}}{L(-P-Q)} \longrightarrow L \longrightarrow \underset{\substack{\| \\ L \otimes k(P) \oplus L \otimes k(Q)}}{k(P) \oplus k(Q)} \longrightarrow 0.$$

As before, $h^1(L(-P-Q)) = 0$ and hence arguing as above, the map

$$H^0(C, L) \otimes_k \mathcal{O}_C \longrightarrow L \otimes k(P) \oplus L \otimes k(Q)$$

is surjective. This means $|D|$ separates points, i.e. one can find a section of L which is zero on P and non zero on Q.

Now let $P = Q$ and let $m_P \subseteq \mathcal{O}_C$ be the maximal ideal sheaf P. We have an exact seq.

$$\begin{array}{ccccccc} 0 \longrightarrow & L(-2P) & \longrightarrow & L & \longrightarrow & \mathcal{O}_C/\mathfrak{m}_P^2 \otimes L & \longrightarrow 0 \\ & \parallel & & \parallel & & \parallel & \\ & \mathcal{O}(D-2P) & & \mathcal{O}(D) & & \mathcal{O}_C/\mathfrak{m}_P^2 & \end{array}$$

We know, since $\deg(L(-2P)) > 2g-2$, that $h^1(L(-2P)) = 0$, whence we have a surjection

$$H^0(C, L) \twoheadrightarrow H^0(C, \mathcal{O}_C/\mathfrak{m}_P^2) = \mathcal{O}_{C,P}/\mathfrak{m}_P^2 \cong k \oplus \mathfrak{m}_P/\mathfrak{m}_P^2.$$

Hence given a non-zero element of $\mathfrak{m}_P/\mathfrak{m}_P^2$, say the image of π, where $\pi \in \mathfrak{m}_P$ is a uniformizing parameter for $\mathcal{O}_{C,P}$, we can find a section $s \in H^0(C, L)$ whose image in k is zero and whose image in $\mathfrak{m}_P/\mathfrak{m}_P^2 \neq 0$, i.e., $s(P) = 0$, $s(\vec{v}) \neq 0$, where $\vec{v}$ is any non-zero tangent vector at P.

This means the map $C \longrightarrow \mathbb{P}(V)$, $V = \Gamma(C, L)^*$, is an embedding.

$$\boxed{\text{If } \deg L > 2g \implies \text{the map } C \longrightarrow \mathbb{P}(V),\ V = \Gamma(C,L)^* \text{ is an embedding.}}$$ — (3)

Remark: This shows that every smooth complete curve is projective, and every effective divisor is ample. In fact every divisor of positive degree is ample. In other words every line bundle of +ve degree is ample.