Feb 9, 2021

Lecture 10

$k = \bar{k}$, A abelian var$/k$

Recall we showed that if L a line bundle on A with $H^0(L) \neq 0$, then L is ample if and only if $K(L)$ is finite.

A slightly simpler proof for the "only if" part.

We know that $L|_{K(L)^0}$ is trivial. On the other hand, since L is ample on A, $L|_{K(L)^0}$ is ample on $K(L)^0$. The only way this can happen is if $K(L)^0$ is a point. //

We stated but did not prove the following theorem:

Theorem: Let X and Y be complete varieties, Z a connected reduced scheme, and L a line bundle on $X \times_k Y \times_k Z$ such that its restrictions to $\{x_0\} \times_k Y \times_k Z$, $X \times_k \{y_0\} \times_k Z$, and $X \times_k Y \times_k \{z_0\}$ are all trivial for some $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. Then L is trivial.

Remarks: By a scheme we mean a finite type scheme over k. A point is always a closed point, whence a k-rational point.

Proof:

Suppose first that X is a non-singular (= smooth) curve. Let $J(X)$ (= $\mathrm{Pic}^0(X)$) be its Jacobian variety[1]. $J(X)$ has (and is defined by) the following universal property:

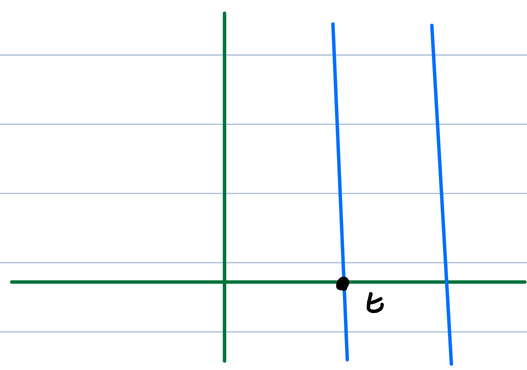
$\longrightarrow$ P.T.O.

[1] We will construct Jacobians of curves later in the course.

There exists a line bundle $\mathcal{L}_u$ on $X \times_k J(A)$ such that: whenever T is a scheme and we have a family $\mathcal{L}$ of line bundles on X parameterised by T, i.e. $\mathcal{L}$ is a line bundle on $X_T := X \times_k T$, then there exists a unique map of schemes

$$\tau : T \longrightarrow J(A)$$

such that $(1 \times \tau)^* \mathcal{L}_u \cong \mathcal{L} \otimes p_2^* \mathcal{M}$ for some line bundle $\mathcal{M}$ on T.



We regard L (the line bundle of the statement of the theorem) as a family of line bundles on X parameterised by $Y \times Z$. From the remarks about the Jacobian, this means we have a map

$$Y \times Z \xrightarrow{\phi} J(A)$$

such the pull back of $\mathcal{L}_u$ to $X \times Y \times Z$ is "essentially" L. We know that $L|_{X \times Y \times \{z_0\}}$ is trivial. This means that $L|_{X \times \{y\} \times \{z_0\}}$ is trivial for every $y \in Y$.

Hence $\phi(y, z_0) = 0$ $\forall y \in Y$, where $0 \in J(A)$ represents the trivial line bundle on X $\left(\begin{array}{c} x \;\; \mathcal{O}_X \\ \downarrow \\ \mathrm{Spec}\,k \end{array} \Rightarrow \mathrm{Spec}\,k \longrightarrow J(A) \right)$

Therefore, since Y is complete, we get a map from $\tilde{\phi} : Z \longrightarrow J(A)$ such that $\phi = \tilde{\phi} \circ p_Z$, where p_Z is the projection $Y \times Z \longrightarrow Z$.

$$\begin{tikzcd}
Y \times Z \arrow[d, "p_Z"'] \arrow[dr, "\phi"] \arrow[dr, phantom, "\circlearrowleft", near start, shift right=3ex] & \\
Z \arrow[r, "\tilde{\phi}"'] & J(A)
\end{tikzcd}$$

Now $L|_{\{y_0\} \times Z}$ is trivial, which means that

$$\phi(y_0, z) = 0 \quad \forall \; z \in Z$$

i.e. $\quad \tilde{\phi}(z) = 0 \qquad \forall \; z \in Z.$

So the map $Y \times Z \longrightarrow J(A)$ is the constant map zero. This means that $L|_{X \times \{y\} \times \{z\}}$ is trivial $\forall \; (y, z) \in Y \times Z$. Hence

$$L \cong p_X^* \mathcal{M}$$

for some line bundle $\mathcal{M}$ on $Y \times Z$. However,

$$\mathcal{M} = p_X^* \mathcal{M}|_{\{x_0\} \times Y \times Z} = L|_{\{x_0\} \times Y \times Z} \underset{\substack{\uparrow \\ \text{by hypothesis.}}}{=} \mathcal{O}_{Y \times Z}$$

Hence $\mathcal{M}$ is trivial, and therefore L is trivial.

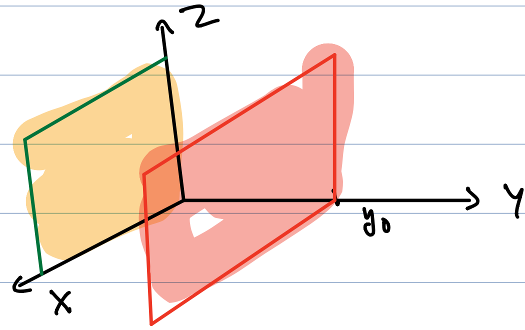
This proves the theorem when X is a smooth curve. Let us now move to the general case. Let $x \in X$. It is a well known fact (since is a complete variety) that there is an irreducible curve C in X which connects x with x_0. Indeed, if X is projective using hyperplane sections, Bertini, and induction on $\dim X$, we are done. If X is not projective one uses Chow's Lemma which says there is a projective variety $\tilde{X}$ and a birational (proper) map $\tilde{X} \xrightarrow{\pi} X$ of k-varieties.

To resume, we have $C \hookrightarrow X$, a curve such that $x_0, x \in C$. Let $\tilde{C} \xrightarrow{f} C$ be the normalisation of C. From from what we have proved, $(f \times 1_Y \times 1_Z)^* L$ is trivial on $\tilde{C} \times Y \times Z$. This shows that $L|_{\{x\} \times Y \times Z}$ is trivial. In particular $L|_{\{x\} \times Y \times \{z\}}$ is trivial for all $(x, z) \in X \times Z$. Since Y is a complete variety (and regarding L as family of line bundles on Y parametrised by $X \times Z$) we see that

$$L \simeq p_{13}^* M$$

for some line bundle M on $X \times Z$ ($p_{13} : X \times Y \times Z \longrightarrow X \times Z$, obvious projection). Now

$$M \cong p_{13}^* M \Big|_{X \times \{y_0\} \times Z} \cong L \Big|_{X \times \{y_0\} \times Z} \underset{\substack{\uparrow \\ \text{by hypothesis}}}{\cong} \mathcal{O}_{X \times Z}$$



Hence L is trivial.

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Facts about smooth complete curves: Let L be a l.b. on a smooth complete curve X.

1. If $\deg L > 2g-2$ then $H^1(X, L) = 0$. [Reason: Use Serre duality and note $h^1(L) = h^0(\omega_X \otimes L^{-1})$ and $\deg(\omega_X \otimes L^{-1}) = 2g-2-\deg L$ and if $\deg L > 2g-2$, this is negative, whence $H^0(X, \omega_X \otimes L^{-1}) = 0$.]
2. If $\deg L > 2g-1$ then L is generated by global sections. In classical terms, if D is a divisor with $\deg D > 2g-1$, then $|D|$ is base point free. (This uses 1.)
3. If $\deg L > 2g$ then L is very ample.