Jan 5, 2021

Lecture 1

1. Abelian varieties are commutative.
2. The notion of a dual abelian variety
3. The "kernel" of a homomorphism of group schemes.
4. Pic^0 of an abelian variety.
5. Criterion for ampleness on an abelian variety.
6. Isogenies.
7. Cohomology of line bundles on an abelian variety.
8. Over $\mathbb{C}$, abelian varieties are tori: $\mathbb{C}^g/\Lambda$.

Reminders.

A variety over a field k is a separated, reduced, irreducible k-scheme of finite type.

The term "geometric" as in "geometrically reduced" etc, means that the property persists after a base change to an alg. closed field.

More precisely, let k be a field and $X \longrightarrow \mathrm{Spec}\, k$ a finite type scheme over k. We say X is geometrically "P" if it has property P and its base change $X \times_k \mathrm{Spec}\, \bar{k}$ ← alg. closure of k, also has property P.

Example: A variety which is regular (i.e. all its local rings $\mathcal{O}_{X,x}$ are regular local rings) need not be geometrically regular.

Let $k \longrightarrow L$ be a purely inseparable finite extension (e.g. $\mathbb{Z}/p\mathbb{Z}(X) \longrightarrow \mathbb{Z}/p\mathbb{Z}(X^{1/p})$).

Then it is easy to see that $L \otimes_k \bar{k}$ is an artin local ring which has nilpotents

alg. closure

$\mathrm{Spec}(L) \longrightarrow \mathrm{Spec}\,k$ variety.

This is regular but not geometrically regular.

Definition: A k-variety (k a field) is said to be smooth if it is geometrically regular. A map $f: X \longrightarrow Y$ of schemes is smooth if it is of finite type (locally finite type?), flat, and all its fibres are smooth.

Abelian varieties over an an alg closed field k:

Let k be an algebraically closed field. An abelian variety A over k is group variety over k such that A is complete over k.

Group variety: G is a group variety if it is a variety and we have a map of varieties:

$$G \times_k G \xrightarrow{m} G$$

such that the group operations are maps of varieties

Definition: A k-variety X is complete if $X \to \mathrm{Spec}\,k$ is proper. Recall, this means $X \to \operatorname{Spec} k$ is separated and universally closed ($X \times_k T \longrightarrow T$ is closed $\forall$ k-schemes T)

The theorem of the square:

Theorem: Let k be an algebraically closed field, X, Y, and Z varieties with X complete. Let

$$f: X \times_k Y \longrightarrow Z$$

be a morphism of varieties such that $X \times \{y_0\}$ maps to a single point z_0 for some $y_0 \in Y(k)$. Then there exists a map of varieties

$$g: Y \longrightarrow Z$$

such that

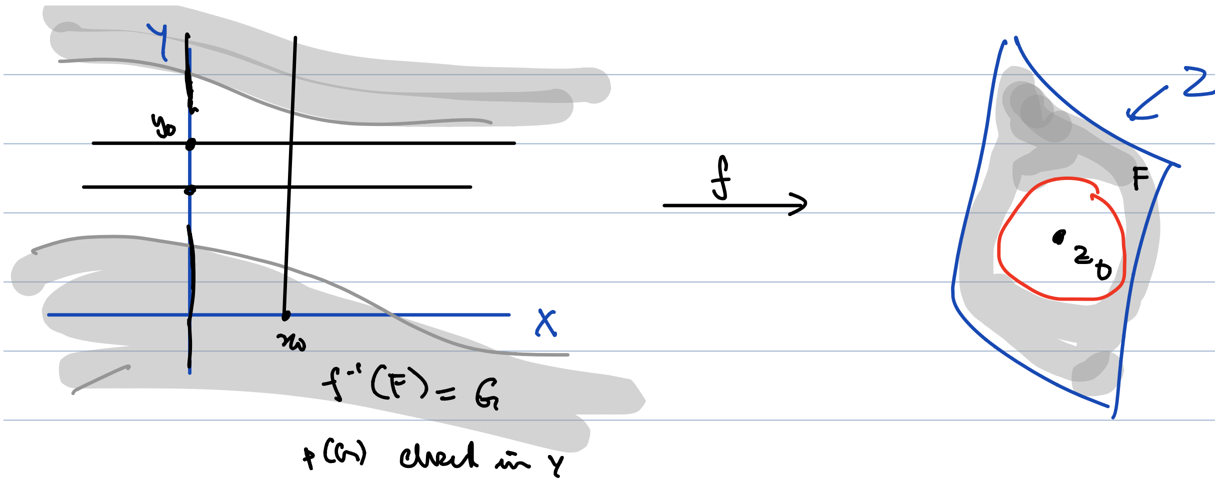
$$f = g \circ p$$

where $p: X \times_k Y \longrightarrow Y$ is the projection.

Proof:

Pick an affine open set U in Z containing $z_0 = f(y_0)$. Let $F = Z \setminus U$. Then F is closed in Z, and hence so is $G = f^{-1}(F)$. Let $H = p(G)$. Then, since X is complete, H is closed in Y. Moreover, a little thought shows $y_0 \notin H$, and hence $V := Y - H$ is a non-empty open set in Y. Note that if $y \in V$, then $X \times \{y\}$ maps to U.

The image $f(X \times \{y\})$ is connected and complete and closed in U. Since U is affine $f(X \times \{y\})$ is affine.



This means $f(X \times \{y\})$ is a point.

Pick any point $x_0 \in X$. Define

$$g : Y \longrightarrow Z$$

by

$$g(y) = f(x_0, y).$$

Clearly, on $p^{-1}(V)$, the two functions f and $g \circ p$ agree. Since $X \times_k Y$ is variety $p^{-1}(V)$ is dense and hence $f = g \circ p$.