HW 5

Due on April 4, 2020 (via moodle by 11:59 pm).

For a commutative ring A, Mod_A denotes the category of A-modules.

For a topological space X, $\mathcal{Psh}(X)$ will denote the category of pre-sheaves on X, and $\mathcal{Sh}(X)$ the category of sheaves on X.

For any scheme Z, $\mathcal{A}(Z)$ will denote the category of $\mathscr{O}_Z$-modules, Z_{qc} the category of quasi-coherent $\mathscr{O}_Z$-modules, and Z_c the category of coherent $\mathscr{O}_Z$-modules. The category of Z-schemes will be denotes $\mathbb{S}\mathrm{ch}_{/Z}$.

The category of abelian groups will be denoted by the symbol $\mathcal{Ab}$.

In the problems that follow, for simplicity we assume all schemes/rings mentioned are noetherian, unless we specifically say something otherwise. Rings mean commutative rings with multiplicative identity 1, and modules are unital (i.e. $1 \cdot x = x$ for elements x in the module).

Group schemes. Fix a scheme S. Let $T \to S$ an S-scheme. We often simply write $T \in \mathbb{S}\mathrm{ch}_{/S}$, rather than $\begin{smallmatrix} T \\ \downarrow \\ S \end{smallmatrix} \in \mathbb{S}\mathrm{ch}_{/S}$. If we wish to emphasise the base scheme S, we will write $T/S \in \mathbb{S}\mathrm{ch}_{/S}$ for typographical convenience. The functor of points $\mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(-, T/S)$ will be denoted $h_{T/S}$. Let $W \in \mathbb{S}\mathrm{ch}_{/T}$. Then it is easy to see that we have a bijection of sets

$$(\dagger) \qquad h_{X_T/T}(W/T) \xrightarrow{\sim} h_{X/S}(W/S)$$

In greater detail, given a map of T-schemes $\varphi\colon W \to X_T = X \times_S T$, the map $p_1 \circ \varphi$ gives us an map of S-schemes $W \to X$. Conversely, suppose $w_0\colon W \to T$ is the structure map of W as a T scheme. Let $w\colon W \to S$ be any map of S-schemes, where we now regard W as an S-scheme via the forgetful functor. We then have a map of T-schemes $(w, w_0)\colon W \to X \times_S T = X_T$. The two processes, $\varphi \mapsto p_1 \circ \varphi$ and $w \mapsto (w, w_0)$, are inverses of each other. It is not hard to see that the bijection $(\dagger)$ is functorial in W/T. Moreover, if $w_0\colon W \to T$ factors as $W \to T' \to T$, then the resulting bijection $h_{X_{T'}/T'}(W/T') \xrightarrow{\sim} h_{X/S}(W/S)$ is the composite $h_{X_{T'}/T'}(W/T') \xrightarrow{\sim} h_{X_T/T}(W/T) \xrightarrow{\sim} h_{X/S}(W/S)$. Feel free to use these facts in the exercises that follow.

1. Let G be a group scheme over S, i.e. group object in $\mathbb{S}\mathrm{ch}_{/S}$. Let $g\colon Z \to G$ be a map of S-schemes. We regard g as a Z valued point of G. Let $\rho_g\colon h_{G/S}(Z) \to h_{G/S}(Z)$ be the map given by right multiplication on the group $h_{G/S}(Z)$.

(a) Suppose $z\colon T \to Z$ is a Z-scheme. Let $g(z)\colon T \to G$ be the composite $g(z) = g \circ z$. Show that the diagram

$$\begin{array}{ccc} h_{G/S}(Z) & \xrightarrow{\ \rho_g\ } & h_{G/S}(Z) \\ \Big\downarrow{\scriptstyle \text{via } z} & & \Big\downarrow{\scriptstyle \text{via } z} \\ h_{G/S}(T) & \xrightarrow[\ \rho_{g(z)}\ ]{} & h_{G/S}(T) \end{array}$$

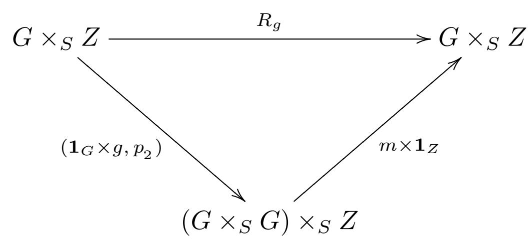
commutes.

(b) Show that $\rho_{g(\cdot)}$ is functorial, i.e. if $F\colon \mathbb{S}\mathrm{ch}_{/Z} \to \mathbb{S}\mathrm{ch}_{/S}$ is the forgetful functor then $\rho_{g(\cdot)}\colon h_{G/S}\circ F \to h_{G/S}\circ F$ is a natural transformation.

(c) Via (†) we can interpet the functorial map $\rho_{g(\cdot)}$ as a natural transformation $\rho_{g(\cdot)}\colon h_{G_Z/Z} \to h_{G_Z/Z}$. Let $R_g\colon G_Z \to G_Z$ be the resulting map (via Yoneda) in $\mathbb{S}\mathrm{ch}_{/Z}$. We will call R_g *the right translation on* G_Z *by* g. Let $m\colon G\times_S G \to G$ be the multiplication map on the group scheme G/S, and $p_2\colon G_Z \to Z$ the natural structure map (which is equal to the second projection on $G\times_S Z$). Show that

$$R_g = (m\circ(\mathbf{1}_G \times g), p_2)$$

where $\mathbf{1_G}$ denotes the identity map $G \xrightarrow{\mathbf{1}_G} G$. In other words, show that the following diagram commutes.



Finite group scheme actions. Let k be an algebraically closed field, and Γ a finite k-algebra (i.e. $\dim_k \Gamma < \infty$) such that $G = \operatorname{Spec}\Gamma$ is a group scheme over k. Let $m\colon G\times_k G \to G$ be the multiplication map, and $m^*\colon \Gamma \to \Gamma\otimes\Gamma$ the corresponding homomorphism of k-algebras. For a k-algebra R, let

$$\Delta_R\colon \Gamma\otimes_k R \to R$$

be the map $\Delta_R(\gamma\otimes r) = \det(\mu(\gamma\otimes r))$, where $\mu(\gamma\otimes r)\colon \Gamma\otimes_k R \to \Gamma\otimes_k R$ is "multiplication by $\gamma\otimes r$". Note that $\Gamma\otimes R$ is a free R-module of finite rank, and hence $\det(\gamma\otimes r)$ makes sense. Δ_R is often called the *norm map* of the R-algebra $\Gamma\otimes_k R$ and $\Delta_R(x)$ is called the norm of x for $x\in\Gamma\otimes_k R$.

Let G act on an affine scheme $X = \operatorname{Spec} B$ of finite type over k, and let the action map be $\nu\colon G\times_k X \to X$. We have a corresponding k-algebra homomorphism $\nu^*\colon B \to \Gamma\otimes_k B$. Let

$$A = \{b\in B \mid \nu^*(b) = 1\otimes b\}.$$

A is often denoted B^G and is called the ring of G-invariants of B.

2. Show that if $f\colon R\to S$ is a k-algebra homomorphism, then $\Delta_S\circ(1_\Gamma\otimes f) = f\circ\Delta_R$.

3. Let $\delta_B\colon B\to B$ be the map given by the formula $\delta_B(\beta) = \Delta_B(\nu^*(\beta))$, $\beta\in B$. Show that $\delta_B(B)\subset A$. (**Caution:** δ_B is not in general an additive homomorphism.)

4. Let G act on $\mathbb{A}^1$ via the trivial action. This induces an action $\widetilde{\nu}$ of G on $X \times_k \mathbb{A}^1$. The corresponding map $\widetilde{\nu}^* \colon B[T] \to \Gamma \times_k B[T]$ of k-algebras is $b \otimes p \mapsto \nu^*(b) \otimes p$, $b \in B$, $p \in k[T]$. As in **3**. we have a map $\delta_{B[T]} \colon B[T] \to B[T]$ given by $\delta_{B[T]} = \Delta_{B[T]} \circ \widetilde{\nu}^*$. For $b \in B$, let $\chi_b(T) = \delta_{B[T]}(T - b)$. Show that the coefficients of χ_b are in A. Show also that b satisfies the polynomial χ_b. **Hint:** Let $\epsilon \colon \Gamma \to k$ be the k-algebra map giving the identity element of $G(k)$ and consider $(\epsilon \otimes 1)(\nu^*(b))$ for any element $b \in B$.

5. Show that A is finitely generated over k and hence B is a finite A-module.

6. Let $Y = \operatorname{Spec} A$ and let $\pi \colon X \to Y$ be the map of schemes induced by the inclusion $A \hookrightarrow B$. Show that π separates the orbits of X under G and as a topological space (Y, π) is the quotient of X by the finite group underlying the group scheme G.

7. Show that

(a) The morphism π of **6**. is finite and surjective.

(b) For every G-invariant morphism $f \colon X \to Z$ of k-schemes, there exists a unique morphism of schemes $g \colon Y \to Z$ such that $f = g \circ \pi$.