HW 4

Due on March 10, 2020 (via moodle by 11:59 pm).

For a commutative ring A, Mod_A denotes the category of A-modules.

For a topological space X, $\mathscr{Psh}(X)$ will denote the category of pre-sheaves on X, and $\mathscr{Sh}(X)$ the category of sheaves on X.

For any scheme Z, $\mathcal{A}(Z)$ will denote the category of $\mathscr{O}_Z$-modules, Z_{qc} the category of quasi-coherent $\mathscr{O}_Z$-modules, and Z_c the category of coherent $\mathscr{O}_Z$-modules.

The category of abelian groups will be denoted by the symbol $\mathscr{Ab}$.

In the problems that follow, for simplicity we assume all schemes/rings mentioned are noetherian, unless we specifically say something otherwise. Rings mean commutative rings with multiplicative identity 1, and modules are unital (i.e. $1 \cdot x = x$ for elements x in the module).

The Grothendieck complex. Let A be a ring, $f\colon X \to \operatorname{Spec} A = T$ a proper map, and $\mathscr{F}$ a coherent sheaf on X, $\mathfrak{U}$ a finite affine open cover of X, and $P^\bullet$ a bounded complex of projective A-modules, with $P^m = 0$ for $m < 0$, such that we have a quasi-isomorphism $\varphi\colon P^\bullet \to \check{C}^\bullet(\mathfrak{U}, \mathscr{F})$. Proposition 12.2 on p.282 of Hartshorne's *Algebraic Geometry* together with of Problem **3** of HW 3 guarantees the existence of such a complex. The complex $P^\bullet$ is called the *Grothendieck complex* of $\mathscr{F}$. As is often the case in informal mathematics, the use of the definite article "the" is of course problematic. Recall that $\varphi \otimes_A M$ is then a quasi-isomorphism for every A-module M, whence $\mathrm{H}^i(P^\bullet \otimes_A M) \xrightarrow{\sim} \mathrm{H}^i(X, \mathscr{F} \otimes_A M)$ (see conventions below) for all i.

Our conventions are as follows. We write X_t for the fibre of f over a point $t \in T$, i.e. $X_t = X \times_T \operatorname{Spec} k(t)$. The right side is often denoted in the literature by the symbol $X \otimes_A k(t)$ and we might indulge in that abuse of notation. For t as above, we write $\mathscr{F}_t$ for the pull back of $\mathscr{F}$ to X_t under the natural map $X_t \to X$. We might also write it as $\mathscr{F}|_{X_t}$. As in earlier homework assignments, $\mathscr{F} \otimes_A M$ is short hand for $\mathscr{F} \otimes_{\mathscr{O}_X} f^*\widetilde{M}$ for any A-module M. If $T' \to T$ is a map of schemes, we write $X_{T'}$ for $X \times_{\operatorname{Spec} A} T'$. The latter scheme will often be written as $X \times_A T'$. The pull back of $\mathscr{F}$ to $X_{T'}$ will be denoted $\mathscr{F}_{T"}$. If $T' = \operatorname{Spec} B$ then we often write X_B or $X \otimes_A B$ for $X_{T'}$. In this case we often write $\mathscr{F}_B$ for $\mathscr{F}_{T'}$.

1. For an A-algebra B, show that $\mathrm{H}^i(P^\bullet \otimes_A B) \xrightarrow{\sim} \mathrm{H}^i(X_B, \mathscr{F}_B)$, for all i. Conclude in particular that $\mathrm{H}^i(P^\bullet \otimes_A k(t)) \xrightarrow{\sim} \mathrm{H}^i(X_t, \mathscr{F}_t)$ for all i.

2. Suppose n is a non-negative integer such that $\mathrm{H}^i(X_t, \mathscr{F}_t) = 0$ for all $i > n$. Show that we may choose our Grothendieck complex $P^\bullet$ so that $P^m = 0$ for $m > n$. **Hint:** Use a technique demonstrated in Lecture 15 to show that $\mathrm{H}^m(P^\bullet) = 0$ for $m > n$, and then use this to modify $P^\bullet$ via a descending induction. **Remark:** Suppose n is a non-negative integer such that $\dim X_t \leq n$ for all $t \in T$.[1] Then the problem above shows, via Theorem 2.7 on p.208 of Hartshorne's *Algebraic*

[1] Here the dimension of a scheme is its dimension as a noetherian topological space (see the definition at the bottom of page 5 of Hartshorne's *Algebraic Geometry*).

Geometry, that we can choose $P^\bullet$ to be such that $P^m = 0$ for $m > n$.

3. Suppose T is connected.

(a) Show that the Euler characteristic $\chi(X_t, \mathscr{F}_t)$ does not depend upon $t \in T$. Recall that the Euler characteristic of any coherent sheaf $\mathscr{G}$ on X_t is the alternating sum $\sum_{i\geq 0}(-1)^i h^i(\mathscr{G})$ where $h^i(\mathscr{G}) = \dim_{k(t)} \mathrm{H}^i(X_t, \mathscr{G})$.

(b) Suppose k is a field and C a complete smooth geometrically connected curve over k (a sufficient condition for this is that X has a k-rational point). Let $X = C_T = C \times_k T$. If $\mathscr{F}$ is a line bundle on X, show that $\deg \mathscr{F}_t$ does not depend upon $t \in T$.

The Jacobian variety of a curve. Here are some definitions and notations. Suppose S is a scheme. We write $\mathbb{S}\mathrm{ch}_{/S}$ for the category of S-schemes. If $S = \operatorname{Spec} A$, we will often write $\mathbb{S}\mathrm{ch}_{/B}$ for $\mathbb{S}\mathrm{ch}_{/S}$.

For a scheme U, $\operatorname{Pic}(U)$ is the set of isomorphism classes of line bundles (i.e. invertible sheaves) on U. It is clear that Pic is a contravariant functor on the category of schemes $\mathbb{S}\mathrm{ch}_{/\mathbf{Z}}$. In fact it is a group functor, since isomorphism classes of line bundles on U form a group under tensor product.

Let S be a scheme and X an S-scheme. As above, let $\mathscr{A}\!\mathscr{b}$ denote the category of abelian groups. The *absolute Picard functor* on X is the functor $\operatorname{Pic}_X\colon \mathbb{S}\mathrm{ch}_{/S} \to \mathscr{A}\!\mathscr{b}$ given by

$$\operatorname{Pic}_X(T) = \operatorname{Pic}(X \times_S T) = \operatorname{Pic}(X_T).$$

The *relative Picard functor* on X is the functor $\mathscr{P}\!ic_{X/S}\colon \mathbb{S}\mathrm{ch}_{/S} \to \mathscr{A}\!\mathscr{b}$ given by

$$\mathscr{P}\!ic_{X/S}(T) = \frac{\operatorname{Pic}(X \times_S T)}{\operatorname{Pic}(T)} = \frac{\operatorname{Pic}_X(T)}{\operatorname{Pic}(T)} \left(= \frac{\operatorname{Pic}(X_T)}{\operatorname{Pic}(T)} \right).$$

If $S = \operatorname{Spec} B$ then we may write $\mathscr{P}\!ic_{X/B}$ for $\mathscr{P}\!ic_{X/S}$.

4. Let X be a smooth complete curve over a field k with a k-rational point D_0 on it.

(a) For $T \in \mathbb{S}\mathrm{ch}_{/k}$ and a line bundle $\mathscr{L}$ on X_T, let $[\mathscr{L}]$ denote the image of $\mathscr{L}$ in $\mathscr{P}\!ic_{X/k}(T)$. Show that if $[\mathscr{L}] = [\mathscr{M}]$ then $\deg \mathscr{L}_t = \deg \mathscr{M}_t$ for all $t \in T$.

(b) For $n \in \mathbf{Z}$, let $\mathscr{P}\!ic^n_{X/k}$ be the functor of $\mathscr{P}\!ic_{X/k}$ given by

$$\mathscr{P}\!ic^n_{X/k}(T) \xrightarrow{\sim} \{[\mathscr{L}] \in \mathscr{P}\!ic_{X/k}(T) \mid \deg \mathscr{L}_t = n,\, t \in T\}.$$

Show that for $n, m \in \mathbf{Z}$, $\mathscr{P}\!ic^n_{X/k} \xrightarrow{\sim} \mathscr{P}\!ic^m_{X/k}$. (**Remark:** If we regard $\mathscr{P}\!ic_{X/k}$ as functor taking values in (**Sets**) rather than $\mathscr{A}\!\mathscr{b}$, for example by applying the forgetful functor, that $\mathscr{P}\!ic^n_{X/k}$ is a subfunctor of $\mathscr{P}\!ic_{X/k}$.)

(c) Show that $\mathscr{P}\!ic^0_{X/k}$ takes values in $\mathscr{A}\!\mathscr{b}$ and is a subfunctor of $\mathscr{P}\!ic_{X/k}$.

(d) Suppose the genus of X is 1. In Lecture 15, we essentially proved that $\mathscr{P}\!ic^1_{X/k}$ is representable by $(X, [\mathscr{O}_{X\times_k X}(\Delta)])$ where Δ is the diagonal in $X \times_k X$. This means, by part (b), $\mathscr{P}\!ic^0_{X/k}$ is representable by X and a suitable element of $\mathscr{P}\!ic^0_{X/k}(X)$. In particular X is a group variety, whence an abelian variety. Show that $3D_0$ gives an embedding of X into the projective plane and that three points in $X(k)$ are collinear if and only if their sum is 0.