HW 3

Due on Jan 25, 2020 (via moodle by 11:59 pm).

For a commutative ring A, Mod_A denotes the category of A-modules.

For a topological space X, $\mathcal{Psh}(X)$ will denote the category of pre-sheaves on X, and $\mathcal{Sh}(X)$ the category of sheaves on X.

For any scheme Z, $\mathcal{A}(Z)$ will denote the category of $\mathscr{O}_Z$-modules, Z_{qc} the category of quasi-coherent $\mathscr{O}_Z$-modules, and Z_c the category of coherent $\mathscr{O}_Z$-modules.

In the problems that follow, for simplicity we assume all schemes/rings mentioned are noetherian, unless we specifically say something otherwise. Rings mean commutative rings with multiplicative identity 1, and modules are unital (i.e. $1 \cdot x = x$ for elements x in the module).

A complex $C^\bullet$ in an abelian category $\mathscr{A}$ is *bounded* if there exist $l, t \in \mathbf{Z}$ with $l \le t$ such that $C^i = 0$ for $i \notin [l, t]$. It is said to be *bounded above* if there exists $t \in \mathbf{Z}$ such that $C^j = 0$ for $j > t$, and it is *bounded below* if there exists $l \in \mathbf{Z}$ such that $C^j = 0$ for $j < l$. Clearly $C^\bullet$ is bounded if and only if it is bounded above and below.

Flatness. Let A be a ring.

1. Suppose $C^\bullet$ is a bounded exact complex of A-modules, with $C^m = 0$ for $m < l$, and C^m flat for $m > l$. Show that C^l is flat. [**Remark:** You might think, from the way the problem is stated, that one only needs $C^\bullet$ to be bounded below. However, to begin a descending induction, you need $C^\bullet$ to be bounded above too.]

2. Let $\varphi\colon M^\bullet \to F^\bullet$ is a quasi-isomorphism of bounded complexes of A-modules with $M^m = F^m = 0$ for $m < l$ for some integer l. Suppose $F^\bullet$ is a flat complex and M^m is flat for all $m > l$. Show that M^l is flat.

3. Suppose $F^\bullet$ is a bounded complex of flat modules. Assume (for simplicty) that $F^m = 0$ for $m < 0$. Let $\varphi\colon P^\bullet \to F^\bullet$ be a quasi-isomorphism of flat complexes, with $P^\bullet$ bounded above. Show that there is a flat complex $Q^\bullet$ with $Q^m = 0$ for $m < 0$, $Q^m = P^m$ for $m > 0$, and a quasi-isomorphism $\psi\colon Q^\bullet \to F^\bullet$ such that $\psi^m = \varphi^m$ for $m > 0$. [**Hint:** Let $K = \ker(\partial_P^0) \cap \ker(\varphi)$ where $\partial_P^m\colon P^m \to P^{m+1}$, $m \in \mathbf{Z}$ are the coboundary maps on $P^\bullet$. Set $Q^0 = P^0/K$.]

4. $P^\bullet$ be a bounded complex of finitely generated projective A-modules with $P^m = 0$ for $m < 0$.

(a) Show that there exists a finitely generated module Q such that we have a functorial isomorphism $\mathrm{H}^0(P^\bullet \otimes_A M) \xrightarrow{\sim} \mathrm{Hom}_A(Q, M)$ for all $M \in \mathrm{Mod}_A$. [**Hint:** Q is the cokernel of the transpose of a certain map of projective modules.]

(b) Show that Q as above is projective if and only if the natural map $\mathrm{H}^0(P^\bullet) \otimes_A M \to \mathrm{H}^0(P^\bullet \otimes M)$ is an isomorphism for all $M \in \mathrm{Mod}_A$. Show also that this happens if and only $\mathrm{H}^0(P^\bullet)$ projective.