HW 2

Due on Jan 20, 2020 (via moodle by 2 pm).

For a commutative ring A, Mod_A denotes the category of A-modules.

For a topological space X, $\mathcal{Psh}(X)$ will denote the category of pre-sheaves on X, and $\mathcal{Sh}(X)$ the category of sheaves on X.

For any scheme Z, $\mathcal{A}(Z)$ will denote the category of $\mathscr{O}_Z$-modules, Z_{qc} the category of quasi-coherent $\mathscr{O}_Z$-modules, and Z_c the category of coherent $\mathscr{O}_Z$-modules.

One important fact that is always useful when one works with noetherian hypotheses: If Z is a noetherian scheme and $\mathscr{I}$ is an injective object in Z_{qc}, then it is an injective object in $\mathcal{A}(Z)$, and hence is *flasque*, i.e. given an open set U of Z, the restriction map $\mathscr{I}(Z) \to \mathscr{I}(U)$ is surjective. Flasque sheaves are acyclic (as we will see in another set of HW problems later in the course) for the global sections functor $\Gamma(Z, -)$, and hence are enough to compute cohomology of sheaves. The last half sentence means the following: If $\mathscr{F} \in Z_{qc}$, with Z noetherian, then $\mathrm{H}^i(Z, \mathscr{F})$ can be computed using the formula $\mathrm{H}^i(Z, \mathscr{G}) = \mathrm{H}^i(\Gamma(Z, \mathscr{I}^\bullet))$, where $\mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$ in the category Z_{qc}. We point out that an injective object in Z_{qc} need not be an injective object in $\mathcal{Sh}(Z)$, and therefore the previous statement is significant.

Quasi-compact and quasi-separated maps. Some definitions (not all of which are in Hartshorne) are in order. A topological space is *quasi-compact* if every open cover has a finite subcover. An affine scheme is always quasi-compact. If such a space is Hausdorff it is called *compact*. A map of $f\colon X \to Y$ spaces is *quasi-compact* if there is a cover $\{V_\alpha\}$ of Y by affine open subschemes such that $f^{-1}(V_\alpha)$ is quasi-compact for all indices α. A map of schemes $f\colon X \to Y$ is *quasi-separated* if the diagonal map $\Delta_{X/Y}\colon X \to X \times_Y X$ is quasi-compact. A scheme X is quasi-separated if the canonical map $X \to \mathrm{Spec}\,\mathbf{Z}$ is quasi-separated,. A separated map is quasi-separated. It is worth emphasising that all these properties just describes are *stable under arbitrary base changes.*

In Problem **3**. you may use the fact that if $f\colon X \to Y$ is a quasi-compact, quasi-separated map of schemes, then the higher direct images $\mathrm{R}^i f_*\mathscr{F}$ of a quasi-coherent sheaf $\mathscr{F}$ are quasi-coherent. You may also assume that if a scheme X is quasi-compact and quasi-separated, and U is a quasi-compact open subscheme of X, then a given $\mathscr{G} \in U_{qc}$, there exists $\mathscr{F} \in X_{qc}$ such that $\mathscr{G} = \mathscr{F}|_U$.

Higher direct images. Recall that if $f\colon X \to Y$ is a continuous map between topological spaces, $f_*\colon \mathcal{Sh}(X) \to \mathcal{Sh}(Y)$ is the functor described by $f_*(\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for every open set V in Y, where $\mathscr{F} \in \mathcal{Sh}(X)$. The functor f_* is called the *direct image functor of* f and $f_*\mathscr{F}$ the *direct image of* $\mathscr{F}$. It is easy to see that (a) in fact $f_*(\mathscr{F}) \in \mathcal{Sh}(X)$ (a priori it lies in $\mathcal{Psh}(X)$), and (b) f_* is left exact. The *higher direct images* $\mathrm{R}^i f_*$*, $i \geq 0$, of* f are the right derived functors of f_*. In other words, for $\mathscr{F} \in \mathcal{Sh}(X)$, and $i \geq 0$, $\mathrm{R}^i f_*\mathscr{F} = \mathrm{H}^i(f_*\mathscr{I}^\bullet)$, where $\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$.

1. Let $f\colon X \to Y = \operatorname{Spec} A$ be a quasi-compact, quasi-separated map of noetherian schemes (note that Y is affine). Let $\mathscr{F} \in X_{qc}$. Show, without using spectral sequences, that the sheaf on Y associated to the A-module $\mathrm{H}^i(X, \mathscr{F})$ is $\mathrm{R}^i f_*\mathscr{F}$ for all i. This amounts to showing that $\Gamma(Y, \mathrm{R}^i f_*\mathscr{F})) = \mathrm{H}^i(X, \mathscr{F})$. [**Hint:** Use the fact that $\Gamma(Y, -)\colon X_{qc} \to \mathrm{Mod}_A$ is an equivalence of categories, since Y is affine. The pseudo-inverse to this functor is $\widetilde{(-)}$.]

Affine maps. A map of schemes $h\colon X \to Y$ is said to be *affine* if $h^{-1}(V)$ is an affine open subscheme of X for every affine open subscheme V of Y.

2. Show:
(a) A closed immersion of schemes is an affine map.
(b) An affine map is quasi-compact and separated.
(c) A separated map is quasi-separated.

3. Show that the following are equivalent for a map of schemes $h\colon X \to Y$, where X is quasi-compact and quasi-separated.
(a) h is affine.
(b) $h_*\colon X_{qc} \to \mathcal{A}(Y)$ is exact.
(c) $\mathrm{R}^i h_*\colon X_{qc} \to \mathcal{A}(Y)$ is zero for $i \geq 1$.

4. Part (b) below may not be so well known to you. Use the fact that under the hypotheses of that problem, the natural map $X \times_S X \to X \times_{\mathbf{Z}} X$ is a closed immersion. And to see the latter, note that the just mentioned natural map is the base change of the map $S = S \times_S S \to S \times_{\mathbf{Z}} S$ via the map $X \times_{\mathbf{Z}} X \to S \times_{\mathbf{Z}} S$ (details are left to you).
(a) Let X be a separated scheme. Show that the intersection of two affine open subschemes of X is again an affine open subscheme.
(b) Suppose $f\colon X \to S$ is separated, with S a separated scheme. Let $U = \operatorname{Spec} A$ be an affine open subscheme of X with $i\colon U \to X$ the open immersion. Show that i is affine.

The box tensor product $\boxtimes$. Let k be a field, and X and Y k-schemes. Consider the cartesian square (the square box at the centre of the square is a way of indicating that the square is cartesian):

$$(*)\qquad \begin{array}{ccc} X \times_k Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & \square & \downarrow \\ X & \longrightarrow & \operatorname{Spec} k \end{array}$$

Let $\mathscr{F} \in X_{qc}$ and $\mathscr{G} \in Y_{qc}$. Define the box tensor product $\mathscr{F} \boxtimes \mathscr{G}$ of $\mathscr{F}$ and $\mathscr{G}$ by the formula

$$(1)\qquad \mathscr{F} \boxtimes \mathscr{G} := p_1^*\mathscr{F} \otimes_{\mathscr{O}_{X \times_k Y}} p_2^*\mathscr{G}.$$

Let X, Y, $\mathscr{F}$, and $\mathscr{G}$ be as above for the problems this section.

5. Suppose X and Y are affine schemes, and suppose $M = \Gamma(X, \mathscr{F})$, $N = \Gamma(Y, \mathscr{G})$. Show that $\mathscr{F} \boxtimes \mathscr{G} = \widetilde{M \otimes_k N}$.

6. Suppose$\mathscr{F} \to \mathscr{C}^\bullet$ and $\mathscr{G} \to \mathscr{D}^\bullet$ are resolutions of $\mathscr{F}$ and $\mathscr{G}$ in X_{qc} and Y_{qc} respectively. Show

(a) $\mathscr{F}\boxtimes\mathscr{G}\to\mathscr{F}\boxtimes\mathscr{D}^\bullet$ is a resolution of $\mathscr{F}\boxtimes\mathscr{G}$.
(b) $\mathscr{F}\boxtimes\mathscr{G}\to\mathscr{C}^\bullet\boxtimes\mathscr{D}^\bullet$ is a resolution of $\mathscr{F}\boxtimes\mathscr{G}$.
[**Hint:** Reduce to X and Y affine. Then use results about double complexes from the previous HW.]

Čech complexes and the box tensor product. In the problems below, we use the following notation. Let Z be a scheme, U an open subscheme of Z, and $i\colon U\to Z$ the resulting open immersion. For $\mathscr{H}\in Z_{qc}$ we write ${}_U\mathscr{H}$ for the direct image under i of $\mathscr{H}|_U$. Thus

$${}_U\mathscr{H}=i_*i^*\mathscr{H}.$$

For an open cover $\mathfrak{U}=\{U_\alpha\}$ of Z, and and sequence of indices $\boldsymbol{\mu}=(\mu_0,\dots,\mu_p)$ from the index set of the cover $\mathfrak{U}$, set $U_{\boldsymbol{\mu}}=U_{\mu_0}\cap\dots\cap U_{\mu_p}$, and $|\boldsymbol{\mu}|=p$. It is then clear that the p^{th} term of the sheaf Čech complex $\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{H})$ is given by

$$\mathscr{C}^p(\mathfrak{U},\,\mathscr{H})=\bigoplus_{|\boldsymbol{\mu}|=p}{}_{U_{\boldsymbol{\mu}}}\mathscr{H}.$$

Hypotheses: We will be assuming that we are given the cartesian diagram $(*)$ above. We further assume $X\to\operatorname{Spec}k$ and $Y\to\operatorname{Spec}k$ are separated and of finite type (though, everything goes through if you just assume they are separated and quasi-compact). The finite type hypothesis ensures that two these k-schemes are noetherian.

Fix open covers $\mathfrak{U}$ of X and $\mathfrak{V}$ of Y. In what follows, as before $\mathscr{F}\in X_{qc}$, $\mathscr{G}\in Y_{qc}$.

7. If $\mathfrak{U}$ consists affine open subschemes of X, show that $\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})$ is a $\Gamma(X,\,-)$-acyclic resolution of $\mathscr{F}$. Conclude that the natural maps $\check{\mathrm{H}}^n(\mathfrak{U},\,\mathscr{F})\to\mathrm{H}^n(X,\,\mathscr{F})$ is an isomorphism for every n.

8. Show that $\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})\boxtimes\mathscr{C}^\bullet(\mathfrak{V},\,\mathscr{G})$ is a resolution of $\mathscr{F}\boxtimes\mathscr{G}$.

9. Suppose $\mathfrak{U}$ and $\mathfrak{V}$ consist of affine open subschemes of X and Y respectively.
(a) Show that $\Gamma(X\times_k Y,\,\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})\boxtimes\mathscr{C}^\bullet(\mathfrak{V},\,\mathscr{G}))=C^\bullet(\mathfrak{U},\,\mathscr{F})\otimes_k C^\bullet(\mathfrak{V},\,\mathscr{G})$.
[**Hint:** It may be easier to first show that if U and V are affine open subschemes of X and Y respectively, then $\Gamma(X\times_k Y,\,{}_{U\times_k V}(\mathscr{F}\boxtimes\mathscr{G}))=\mathscr{F}(U)\otimes_k\mathscr{G}(V)$.]
(b) Show that $\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})\boxtimes\mathscr{C}^\bullet(\mathfrak{V},\,\mathscr{G})$ is a $\Gamma(X\times_k Y,\,-)$-acyclic resolution of $\mathscr{F}\boxtimes\mathscr{G}$.

10. Prove the Kunneth formula

$$\mathrm{H}^n(X\times_k Y,\,\mathscr{F}\boxtimes\mathcal{G})\xrightarrow{\ \sim\ }\bigoplus_{i+j=n}\mathrm{H}^i(X,\,\mathscr{F})\otimes_k\mathrm{H}^j(Y,\,\mathscr{G})\qquad(n\geq 0).$$