HW 1

Due on Jan 12, 2020 (via moodle by 2 pm).

Eilenberg-Zilber over fields. The aim of this section is to prove the Eilenberg-Zilber theorem over fields.

Fix a field k. Let $M^\bullet$ and $N^\bullet$ be two bounded below[1] complexes of k-vector spaces. For each integer n, let

$$\mathcal{EZ}_n = \mathcal{EZ}_n(M^\bullet, N^\bullet) \colon \bigoplus_{i+j=n} H^i(M^\bullet) \otimes_k H^j(N^\bullet) \longrightarrow H^n(M^\bullet \otimes_k N^\bullet)$$

be the Eilenberg-Zilber map defined in the supplementary notes. In the problems that follow, you may assume that $\mathcal{EZ}_n$ is bifunctorial in both its arguments for each n.

1. Let

$$0 \longrightarrow M'^\bullet \longrightarrow M^\bullet \longrightarrow M''^\bullet \longrightarrow 0$$

be a short exact sequence of bounded below complexes of k-vector spaces.

(a) For a complex of vector spaces $C^\bullet$, write $P^n(C^\bullet)$ for $H^n(C^\bullet \otimes_k N^\bullet)$. Show that there is a long exact sequence of vector spaces

$$\ldots \longrightarrow P^n(M'^\bullet) \longrightarrow P^n(M^\bullet) \longrightarrow P^n(M''^\bullet) \longrightarrow P^{n+1}(M'^\bullet) \longrightarrow \ldots.$$

(b) For a complex of vector spaces $C^\bullet$, write $Q^n(C^\bullet)$ for the direct sum of vector spaces $\bigoplus_{i+j=n} H^i(C^\bullet) \otimes_k H^j(N^\bullet)$. Show that there is a long exact sequence

$$\ldots \longrightarrow Q^n(M'^\bullet) \longrightarrow Q^n(M^\bullet) \longrightarrow Q^n(M''^\bullet) \longrightarrow Q^{n+1}(M'^\bullet) \longrightarrow \ldots.$$

2. Show that if $M^\bullet$ is concentrated in one degree, i.e. there exists an integer r such that $M^i = 0$ for $i \neq r$, then $\mathcal{EZ}_n$ is an isomorphism for all n.

3. Consider the exact sequence of complexes in problem **1**. Show that if for any two of $M'^\bullet$, $M^\bullet$, $M''^\bullet$, the Eilenberg-Zilber maps are isomorphisms for all $n \in \mathbb{Z}$, then they are so for the third complex too (the complex $N^\bullet$ remains fixed for this problem). [**Hint:** It is easy to see that the the Eilenberg-Zilber maps are compatible with the long exact sequences in problem **1**. You may assume this.]

4. Show that $\mathcal{EZ}_n(M^\bullet, N^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$ in the following way. First show that it is enough to assume $M^\bullet$ is bounded above (it is already bounded below by our hypothesis). Next assume without loss of generality that $M^i = 0$ for $i < 0$. Let $\ell(M^\bullet)$ be the largest index p such that $M^p \neq 0$ (if $M^p = 0$ for all p, set $\ell(M^\bullet) = -1$). Use induction on $\ell(M^\bullet)$. Note that if $M^\bullet$ is a non-zero complex, then with $\ell = \ell(M^\bullet)$, $M^\ell[-\ell]$ is a subcomplex of $M^\bullet$, where $M^\ell[-\ell]$ is the complex which in degree ℓ is M^ℓ and is zero in all other degrees. Now use problem **3**.

[1] This hypothesis can be relaxed, but we keep it for simplicty, and this the only case we need

Double complexes. You will need to look at the notes on double complexes on the course website for these problems. Let $\mathscr{A}$ be an abelian category.[2] Fix a complex $(\mathcal{K}^\bullet, d)$ which is the total complex of an *anti-commuting* double complex $K^{\bullet\bullet} = (K, d_1, d_2)$ with the property that $K^{\bullet\bullet}$ is bounded on the left and below by p_0 and q_0 respectively. Recall that under the last hypothesis, $\mathcal{K}^\bullet = {}'\mathrm{Tot}^\bullet K^{\bullet\bullet}$ exists without assuming $\mathscr{A}$ has countable direct sums. These assumptions are valid for the next four problems. for problem **8**, you will need the notion of a mapping cone of a map of complexes and the trivial fact that such a map is a quasi-isomorphism if and only if the mapping cone is exact. Please look up the definition and this property from any source, for example the supplementary note on the course website. It is basic homological algebra.

5. Let $A^{\bullet\bullet} = (A, \partial_1, \partial_2)$ and $D^{\bullet\bullet} = (D, \delta_1, \delta_2)$ be data given by

$$A^{p,q} = D^{p,q} = K^{p,q}$$

and whose partial coboundaries are given by:

$$\partial_1^{p,q} = d_1^{p,q} \qquad \partial_2^{p,q} = (-1)^p d_2^{p,q}$$
$$\delta_1^{p,q} = (-1)^q d_1^{p,q} \qquad \delta_2^{p,q} = d_2^{p,q}.$$

(a) Show that $(\mathrm{Tot}^\bullet A^{\bullet\bullet}, \partial) = (\mathcal{K}^\bullet, d)$.
(b) Show that $\mathrm{Tot}^\bullet D^{\bullet\bullet}$ is isomorphic to $(\mathcal{K}^\bullet, d)$.
(c) There is an obvious notion of the transpose of a double complex (recall, the default assumption is that a double complex is a commuting double complex). Show that the total complex of a double complex is isomorphic to the total complex of its transpose.

6. Suppose $\mathscr{A}$ is the category of left modules over a ring. Show by *chasing elements* that if the columns of $A^{\bullet\bullet}$ are exact (or, what amounts to the same thing, if the columns of $K^{\bullet\bullet}$ are exact) then $\mathcal{K}^\bullet$ is exact.

7. Do the above problem without the assumption made on $\mathscr{A}$ in that problem. In other words, do it without chasing elements. [**Hint:** This is best done as follows. Reduce the problem to the case where $A^{\bullet\bullet}$ is bounded to the right. Next, without loss of generality, assume that all $A^{pq} = 0$ for $p < 0$. Show that the statement is true when $A^{\bullet\bullet}$ consists of exactly one column. Now apply induction on $p(A^{\bullet\bullet})$, where $p = p(A^{\bullet\bullet})$ is the largest integer j such that the j^{th} column is non-zero. If all columns are zero, set $p = -1$. To apply induction you should note that *right most* column of $A^{\bullet\bullet}$ (i.e. the $p(A^{\bullet\bullet})^{\text{th}}$ column) is a sub double complex of $A^{\bullet\bullet}$ and hence its total complex is a subcomplex of $\mathcal{K}^\bullet$. There is then an obvious short exact sequence of complexes, etc., etc.]

8. Let $p_\circ$ be an integer such that p^{th} column of $A^{\bullet\bullet}$ is zero for every $p < p_\circ$. Let $Z^q = \ker(A^{p_\circ, q-p_\circ} \xrightarrow{\partial_1} A^{p_\circ+1, q-p_\circ})$. Note that ∂_2 induces the structure of a complex on $Z^\bullet$.
(a) Show that $Z^\bullet$ is a subcomplex of $\mathcal{K}^\bullet$.

[2] If you don't know what an abelian category is, feel free to regard $\mathscr{A}$ as a category of modules over a ring.

(b) Show that if the rows of $A^{\bullet\bullet}$ are exact at all places except (possibly) at $p_\circ$, then the natural inclusion $Z^\bullet \hookrightarrow \mathcal{K}^\bullet$ is a quasi-isomorphism. [**Hint:** Assume without loss of generality that $p_\circ = 0$. Consider the *augmented* double complex $\widetilde{A}^{\bullet\bullet}$ which is $A^{\bullet\bullet}$ for non-negative columns, and which has one extra column in the $(-1)^{\text{th}}$ place, namely $Z^\bullet$, together with the obvious inclusion $Z^q \subset A^{0,q}$ as the horizontal differential in the $(-1,q)^{\text{th}}$ place. Show that $\mathrm{Tot}(\widetilde{A}^{\bullet\bullet})$ is exact and equal to the mapping cone of the map $Z^\bullet \to \mathcal{K}^\bullet$. And if you don't know mapping cones, you can look up the definition and first properties here.]