QUIZ 4 (ANALYSIS II)

Feb 14, 2020 (in tutorial) Name: Solutions

In what follows, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. The symbol δ_{ij} is the Kronecker symbol and feel free to ask the TA's what it means if you don't know. Finally, $\mathbf{R}_+ = [0, \infty)$.

(1) Let $(V, \| \, \|)$ be a finite dimensional normed linear space over $\mathbf{K}$ and W a vector subspace of V. Show that W is closed in V.

Solutions: We know $W \cong \mathbf{K}^d$ for some d and that all norms on $\mathbf{K}^d$ are equivalent, whence $\mathbf{K}^d$ is complete in all norms on it, since it is so with respect to $\| \, \|_\infty$. It follows that W is complete in V. It is immediate that W is closed. Indeed, if w^* is in the closure of W, we have a sequence $\{w_n\}$ in W which converges to w^*, and since W is complete, w^* must lie in W.

(2) Let $\langle \, , \, \rangle$ be an inner product on a finite dimensional $\mathbf{K}$-vector space V and let $\| \, \| \colon V \to \mathbf{R}_+$ be the usual norm, namely $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$. Let W be a subspace of V and $w_1, \ldots, w_k$ a basis of W such that $\langle w_i, w_j \rangle = \delta_{ij}$ for $i, j \in \{1, \ldots, k\}$. Let $\pi \colon V \to W$ be the map $\pi(v) = \sum_{i=1}^{k} \langle v, w_i \rangle w_i$. Fix $v \in V$ and define $f_v \colon W \to \mathbf{R}_+$ by the formula $f_v(w) = \|w - v\|$. Show that

(a) f_v is continuous on W;

Solutions: This follows from the inequality $|\|w - v\| - \|w' - v\|| \leq \|w - w'\|$. It is immediate that if $\epsilon > 0$ is given, picking $\delta = \epsilon$ we get $|f_v(w) - f_v(w')| < \epsilon$ whenever $\|w - w'\| < \delta$.

(b) f_v attains its minimum at $\pi(v)$ and if $f_v(w) = f_v(\pi(v))$ for some $w \in W$, then $w = \pi(v)$.

Solutions: If $w \in W$ then we claim that

$$\langle v - \pi(v), w \rangle = 0.$$

Indeed

$$\begin{aligned} \langle v - \pi(v), w_j \rangle &= \langle v, w_j \rangle - \sum_{i=1}^{k} \langle \langle v, w_i \rangle w_i, w_j \rangle \\ &= \langle v, w_j \rangle - \langle v, w_j \rangle = 0. \end{aligned}$$

In particular $\langle v - \pi(v),\, \pi(v)\rangle = 0$. For $w \in W$, we therefore have (by an easy calculation):

$$\begin{aligned} f_v(w)^2 &= \langle v - w, v - w\rangle \\ &= \langle v - \pi v, v - \pi(v)\rangle + \langle \pi(v) - w, \pi(v) - w\rangle \\ &= f_v(\pi(v))^2 + \|\pi(v) - w\|^2 \end{aligned}$$

It follows that $f_v(\pi(v)) \leq f_v(w)$ for $w \in W$, and when $w = \pi(v)$ this minimum is attained. Finally, if $f_v(w) = f_v(\pi(v))$ then the above shows that $\langle \pi(v) - w, \pi(v) - w\rangle = 0$, whence $w = \pi(v)$.