QUIZ 3 (ANALYSIS II)

Feb 7, 2020 (in tutorial) Name: Solutions

(1) Let m and n be non-negative integers with $m \leq n$, $d = n - m$, $\mathbf{R}^n = \mathbf{R}^d \times \mathbf{R}^m$ be the usual decomposition, and $U \subset \mathbf{R}^n$ an open neighbourhood of $\mathbf{0} \in \mathbf{R}^n$. Let $\boldsymbol{\varphi}\colon U \to \mathbf{R}^m$ be a continuous map with $\boldsymbol{\varphi}(\mathbf{0}) = \mathbf{0}$, and $\boldsymbol{\psi}\colon U \to \mathbf{R}^n$ the map given by the formula $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}))$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{R}^d \times \mathbf{R}^m = \mathbf{R}^n$. Suppose (a) $\boldsymbol{\psi}$ is one-to-one, (b) $V = \boldsymbol{\psi}(U)$ is open in $\mathbf{R}^n$, and (c) the inverse map $\boldsymbol{\psi}^{-1}\colon V \to U$ is continuous. Show that the equation

$$\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0} \qquad (*)$$

can be "solved implicitly for $\boldsymbol{y}$ as a function of $\boldsymbol{x}$" in a neighbourhood of $\mathbf{0} \in \mathbf{R}^d$, i.e. show that there exists an open subset W in $\mathbf{R}^d$ containing $\mathbf{0}$ and a *continuous* map $\boldsymbol{f}\colon W \to \mathbf{R}^m$ with $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ such that for every $\boldsymbol{x} \in W$, $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \in U$ and $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = \mathbf{0}$. (The function $\boldsymbol{f}$ is called implicit function associated to $(*)$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ the implicit solution of $(*)$.)

Solution: We can find open subsets $W \subset \mathbf{R}^d$ and $W' \subset \mathbf{R}^m$ such that $\mathbf{0} \in W \times W' \subset V$. Let $I = \{1, \ldots, d\}$, $J = \{d+1, \ldots, n\}$ and let $\pi_I\colon \mathbf{K}^n \to \mathbf{K}^d$ and $\pi_J\colon \mathbf{K}^n \to \mathbf{K}^m$ be the usual the projections.[1] Then $\Psi^{-1}|_{W \times W'}\colon W \times W' \to U$ is of the form $(\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}))$, where $F(\boldsymbol{x}, \boldsymbol{y}) = \pi_J(\Psi^{-1}(\boldsymbol{x}, \boldsymbol{y}))$. In other words,

$$\Psi^{-1}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, F(\boldsymbol{x}, \boldsymbol{y})) \qquad (\boldsymbol{x} \in W, \boldsymbol{y} \in W').$$

Let $\boldsymbol{f}\colon W \to U$ be the map given by the fomula $\boldsymbol{f}(\boldsymbol{x}) = F(\boldsymbol{x}, \mathbf{0})$. Now $F\colon W \times W' \to \mathbf{R}^m$ is continuous since it is the composite of continuous maps. Since $\boldsymbol{f} = F \circ i$, where $i\colon W \hookrightarrow W \times W'$ is the inclusion, $\boldsymbol{x} \mapsto (\boldsymbol{x}, \mathbf{0})$, it is clear $\boldsymbol{f}\colon W \to \mathbf{R}^m$ is continuous. Moreover, $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = \Psi^{-1}(\boldsymbol{x}, \mathbf{0}) \in U$. It remains to show that

$$\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = \mathbf{0} \qquad (\boldsymbol{x} \in W). \qquad (**)$$

This is straightforward. Indeed

$$(\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))) = \Psi(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = \Psi(\Psi^{-1}(\boldsymbol{x}, \mathbf{0})) = (\boldsymbol{x}, \mathbf{0}) \quad (\boldsymbol{x} \in W)$$

giving $(**)$. □

[1] See Corollary 2.2.3 of Lecture 3.

(2) Let $(V, \| \|)$ be a finite dimensional normed linear space over $\mathbf{K}$ where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Suppose $A\colon V \to V$ is a linear transformation such that $\|A\|_L < 1$ where $\| \|_L$ is the operator norm. Show that $I - A$ is invertible where I is the identity map on V.

Solution: See Theorem 2.1.1 of Lecture 7