QUIZ 3 (ANALYSIS II)

Feb 7, 2020 (in tutorial) Name: ______________________________

(1) Let m and n be non-negative integers with $m \leq n$, $d = n - m$, $\mathbf{R}^n = \mathbf{R}^d \times \mathbf{R}^m$ be the usual decomposition, and $U \subset \mathbf{R}^n$ an open neighbourhood of $\mathbf{0} \in \mathbf{R}^n$. Let $\boldsymbol{\varphi}\colon U \to \mathbf{R}^m$ be a continuous map with $\boldsymbol{\varphi}(\mathbf{0}) = \mathbf{0}$, and $\boldsymbol{\psi}\colon U \to \mathbf{R}^n$ the map given by the formula $\boldsymbol{\psi}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}))$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{R}^d \times \mathbf{R}^m = \mathbf{R}^n$. Suppose (a) $\boldsymbol{\psi}$ is one-to-one, (b) $V = \boldsymbol{\psi}(U)$ is open in $\mathbf{R}^n$, and (c) the inverse map $\boldsymbol{\psi}^{-1}\colon V \to U$ is continuous. Show that the equation

$$\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0} \qquad (*)$$

can be "solved implicitly for $\boldsymbol{y}$ as a function of $\boldsymbol{x}$" in a neighbourhood of $\mathbf{0} \in \mathbf{R}^d$, i.e. show that there exists an open subset W in $\mathbf{R}^d$ containing $\mathbf{0}$ and a *continuous* map $\boldsymbol{f}\colon W \to \mathbf{R}^m$ with $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ such that for every $\boldsymbol{x} \in W$, $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \in U$ and $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) = \mathbf{0}$. (The function $\boldsymbol{f}$ is called implicit function associated to $(*)$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ the implicit solution of $(*)$.)

(2) Let $(V, \| \|)$ be a finite dimensional normed linear space over $\mathbf{K}$ where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Suppose $A\colon V \to V$ is a linear transformation such that $\|A\|_L < 1$ where $\| \|_L$ is the operator norm. Show that $I - A$ is invertible where I is the identity map on V.

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