QUIZ 2 (ANALYSIS II)

Jan 24, 2020 (in tutorial) Name: Solutions

Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Fix p and q in $[0, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Such p and q are called *Hölder conjugates*. Assume the following inequality:

$$\text{(Young's inequality)} \qquad ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b \geq 0).$$

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbf{K}^n$ define $\|\boldsymbol{x}\|_p$ and $\|\boldsymbol{x}\|_q$ in the usual way, e.g. $\|\boldsymbol{x}\|_p = \left\{\sum_{i=1}^n |x_i|^p\right\}^{\frac{1}{p}}$. This quiz will establish that $\| \, \|_p$ is a norm.

(1) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{K}^n$ and suppose $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$. Show that $\sum_{i=1}^n |a_i b_i| \leq \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q$. [Hint: First assume $\|\boldsymbol{a}\|_p = \|\boldsymbol{b}\|_q = 1$.]

Solution: Suppose first that $\|\boldsymbol{a}\|_p = \|\boldsymbol{b}\|_q = 1$. By Young's inequality we have

$$|a_i b_i| \leq \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q} \qquad (i = 1, \ldots, n).$$

Summing over $i \in \{1, \ldots, n\}$ we get

$$\begin{aligned} \sum_{i=1}^n |a_i b_i| \leq \sum_{i=1}^n \frac{|a_i|^p}{p} + \sum_{i=1}^n \frac{|b_i|^q}{q} &= \frac{\|\boldsymbol{a}\|_p^p}{p} + \frac{\|\boldsymbol{b}\|_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \tag{$*$}$$

Now drop the assumption that $\|\boldsymbol{a}\|_p = \|\boldsymbol{b}\|_q = 1$. If either $\boldsymbol{a}$ or $\boldsymbol{b}$ equals $\mathbf{0}$, the inequality we have to prove it trivially true. So assume neither is $\mathbf{0}$. Let $\boldsymbol{x} = \boldsymbol{a}/\|\boldsymbol{a}\|_p$ and $\boldsymbol{y} = \boldsymbol{b}/\|\boldsymbol{b}\|_q$, and suppose $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$. Note that $\|\boldsymbol{x}\|_p = \|\boldsymbol{y}\|_q = 1$ and so $(*)$ applies, and we have $\sum_{i=1}^n |x_i y_i| \leq 1$, i.e. $\left(\sum_{i=1}^n |a_i b_i|\right)/\|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q \leq 1$ giving the asserted inequality. ☐

(2) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{K}^n$.

(a) Show that

$$\|\boldsymbol{a}+\boldsymbol{b}\|^p \leq \Big(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\Big)\|\boldsymbol{v}\|_q$$

where $\boldsymbol{v} = ((a_1+b_1)^{p-1}, \ldots, (a_n+b_{n-1})^{p-1})$.

Solution: Let $v_i = (a_i+b_i)^{p-1}$ for $i = 1, \ldots, n$, so that $\boldsymbol{v} = (v_1, \ldots, v_n)$. We have

$$\begin{aligned}\|\boldsymbol{a}+\boldsymbol{b}\|_p^p = \sum_{i=1}^n |(a_i+b_i)^p| &= \sum_{i=1}^n |(a_i+b_i)v_i| \\ &\leq \sum_{i=1}^n |a_i v_i| + \sum_{i=1}^n |b_i v_i| \\ &\leq \|\boldsymbol{a}\|_p\|\boldsymbol{v}\|_q + \|\boldsymbol{b}\|_p\|\boldsymbol{v}\|_q \\ &= (\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p)\|\boldsymbol{v}\|_q,\end{aligned}$$

giving the required inequality. □

(b) Show that $\|\boldsymbol{a}+\boldsymbol{b}\|_p \leq \|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p$.

Solution: Let $\boldsymbol{v}$ be as in part (a). Since $\frac{1}{p} + \frac{1}{q} = 1$, it is therefore straightforward to see that $q(p-1) = p$, giving

$$\|\boldsymbol{v}\|_q^q = \sum_{i=1}^n |a_i+b_i|^{(p-1)q} = \sum_{i=1}^n |a_i+b_i|^p = \|\boldsymbol{a}+\boldsymbol{b}\|_p^p.$$

From part (a) we therefore get

$$\|\boldsymbol{a}+\boldsymbol{b}\|_p^p \leq (\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p)\|\boldsymbol{a}+\boldsymbol{b}\|_p^{\frac{p}{q}}. \tag{†}$$

If $\|\boldsymbol{a}+\boldsymbol{b}\|_p = 0$, there is nothing to prove. So assume $\|\boldsymbol{a}+\boldsymbol{b}\|_p$ is non-zero. Then (†) yields

$$\|\boldsymbol{a}+\boldsymbol{b}\|_p^{p-\frac{p}{q}} \leq \|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p.$$

But $p - \frac{p}{q} = p(1-\frac{1}{q}) = p\frac{1}{p} = 1$ and we are done. □