QUIZ 2 (ANALYSIS II)

Jan 24, 2020 (in tutorial) Name: ______________________________

Let $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Fix p and q in $[0, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Such p and q are called *Hölder conjugates*. Assume the following inequality:

(Young's inequality) $$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b \geq 0).$$

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbf{K}^n$ define $\|\boldsymbol{x}\|_p$ and $\|\boldsymbol{x}\|_q$ in the usual way, e.g. $\|\boldsymbol{x}\|_p = \left\{\sum_{i=1}^n |x_i|^p\right\}^{\frac{1}{p}}$. This quiz will establish that $\|\ \|_p$ is a norm.

(1) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{K}^n$ and suppose $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$. Show that $\sum_{i=1}^n |a_i b_i| \leq \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q$. [Hint: First assume $\|\boldsymbol{a}\|_p = \|\boldsymbol{b}\|_q = 1$.]

(2) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{K}^n$.

(a) Show that

$$\|\boldsymbol{a} + \boldsymbol{b}\|^p \leq \Big(\|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p\Big)\|\boldsymbol{v}\|_q$$

where $\boldsymbol{v} = ((a_1 + b_1)^{p-1}, \ldots, (a_n + b_{n-1})^{p-1})$.

(b) Show that $\|\boldsymbol{a} + \boldsymbol{b}\|_p \leq \|\boldsymbol{a}\|_p + \|\boldsymbol{b}\|_p$.

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