Chennai Mathematical Institute
MU1202 Analysis II
Semester 2, 2019-20
Mid-term Exam

Date: Feb 25, 20

(1) Let $(V, \| \ \|)$ be a normed linear space over $\mathbf{K}$. Let $C \subset V$.

(a) (5 marks) When is $v \in V$ called a limit point of C?

Solution: See Definition 1.2.2 of Lecture 1.

(b) (5 marks) Define the closure of C.

Solution: See Definition 1.2.2 of Lecture 1 again.

(2) (10 marks) Show that if $\| \ \|$ is any norm on $\mathbf{R}^n$ then $(\mathbf{R}^n, \| \ \|)$ is complete by showing that $(\mathbf{R}^n, \| \ \|_\infty)$ is complete and then using a theorem (state the theorem you use).

Solution: We will use the theorem that all norms on $\mathbf{R}^n$ are equivalent. Therefore it is enough to show that $(\mathbf{R}^n, \| \ \|_\infty)$ is complete. Recall, $\|(x_1, \dots, x_n)\|_\infty = \max_{1\le i\le n}|x_i|$ for $(x_1, \dots, x_n) \in \mathbf{R}^n$. Suppose $\{\boldsymbol{x}_m\}$ is Cauchy in $(\mathbf{R}^n, \| \ \|_\infty)$. Then, given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\|\boldsymbol{x}_m - \boldsymbol{x}_k\|_\infty < \epsilon/2$ for $m, k \ge N$. For $i \in \{1, \dots, n\}$ and $m \in \mathbf{N}$, let x_{im} be the i^{th} component of $\boldsymbol{x}_m$. Then $|x_{im} - x_{ik}| < \epsilon/2$ for all $m, k \ge N$ and $1 \le i \le n$. It follows that for each fixed $i \in \{1, \dots, n\}$, $\{x_{im}\}$ is Cauchy in $\mathbf{R}$. Since $\mathbf{R}$ is complete, this sequence has a limit, say x_i. Note that

$$|x_{im} - x_i| = |x_{im} - \lim_{k\to\infty} x_{ik}| = \lim_{k\to\infty} |x_{im} - x_{ik}| \le \tfrac{1}{2}\epsilon \qquad (m \ge N, i \in \{1, \dots, n\}).$$

Let $\boldsymbol{x} = (x_1, \dots, x_n)$. Taking the maximum over $i \in \{1, \dots, n\}$ in the above inequality we get

$$\|\boldsymbol{x}_m - \boldsymbol{x}\|_\infty \le \tfrac{1}{2}\epsilon < \epsilon \qquad (m \ge N).$$

This means $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{R}^n, \| \ \|_\infty)$. Thus $(\mathbf{R}^n, \| \ \|_\infty)$ is complete. □

(3) Let (X, d) be a metric space.

(a) (2 marks) Give the definition of a compact subset of X.

Solution: See Definition 1.1.1 of Lecture 4.

(b) (4 marks) Let K be a compact subset of X. Show that if (Y, ϱ) is another metric space and $f\colon X \to Y$ a continuous map, then $f(K)$ is compact.

Solution: See item **1.** of Lecture 7.

(c) (4 marks) Show that the composite of continuous maps between metric spaces is continuous.

Solution: See item **2.** of Lecture 7.

(4) On any Euclidean space $\mathbf{R}^k$ let $B_\infty(\boldsymbol{p}, r)$ denote the open ball centred at $\boldsymbol{p}$ of radius r with respect to $\| \, \|_\infty$. Let $n = d + m$, where d and m are non-negative integers and make the standard identification $\mathbf{R}^n = \mathbf{R}^d \times \mathbf{R}^m$. Let $\pi \colon \mathbf{R}^n \to \mathbf{R}^d$ and $\pi' \colon \mathbf{R}^n \to \mathbf{R}^m$ be the projection maps given by $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$ and $\pi'(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{R}^n$.

(a) (5 marks) Let $B = B_\infty((\boldsymbol{a}, \boldsymbol{b}), r) \subset \mathbf{R}^n$. Show that $\pi(B) = B_\infty(\boldsymbol{a}, r)$ and $\pi'(B) = B_\infty(\boldsymbol{b}, r)$.

Solution: Let $\boldsymbol{a} = (a_1, \dots, a_d)$ and $\boldsymbol{b} = (b_1, \dots, b_m)$. Let $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{R}^n$ with $\boldsymbol{x} = (x_1, \dots, x_d)$ and $\boldsymbol{y} = (y_1, \dots, y_m)$. Then

$$\begin{aligned}(\boldsymbol{x}, \boldsymbol{y}) \in B &\iff \|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{a}, \boldsymbol{b})\|_\infty < r \\ &\iff |x_i - a_i| < r \text{ and } |y_j - b_j| < r \text{ for } i = 1, \dots, d \text{ and } j = 1, \dots, m \\ &\iff \boldsymbol{x} \in B_\infty(\boldsymbol{a}, r) \text{ and } \boldsymbol{y} \in B_\infty(\boldsymbol{b}, r)\end{aligned}$$

The above sequence of equivalences show that $B = B_\infty(\boldsymbol{a}, r) \times B_\infty(\boldsymbol{b}, r)$, whence $\pi(B) = B_\infty(\boldsymbol{a}, r)$ and $\pi'(B) = B_\infty(\boldsymbol{b}, r)$. $\square$

(b) (5 marks) Show, using (a), that if $A \times B$ is open in $\mathbf{R}^n$ where $A \subset \mathbf{R}^d$ and $B \subset \mathbf{R}^m$, then A is open in $\mathbf{R}^d$ and B is open in $\mathbf{R}^m$.

Solution: If U is open in $\mathbf{R}^n$ then $U = \cup_\alpha B_\infty((\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha), r_\alpha)$ from which we get $\pi(U) = \cup_\alpha \pi(B_\infty((\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha), r_\alpha))$. By part (a) this means $\pi(U) = \cup_\alpha(B_\infty(\boldsymbol{x}_\alpha, r_\alpha))$, whence $\pi(U)$ is open in $\mathbf{R}^d$. Similarly $\pi'(U)$ is open in $\mathbf{R}^m$. Since $\pi(A \times B) = A$ and $\pi'(A \times B) = B$, A must be open in $\mathbf{R}^d$ and B in $\mathbf{R}^m$ if $A \times B$ is open in $\mathbf{R}^n$. $\square$

(5) (a) (2 marks) Let U be an open subset of $\mathbf{R}^k$, V an open subset of $\mathbf{R}^n$, and $\boldsymbol{a}$ a point in U. Let $\boldsymbol{g} \colon U \to V$ and $\boldsymbol{f} \colon V \to \mathbf{R}^m$ be maps such that $\boldsymbol{g}$ is differentiable at $\boldsymbol{a}$ and $\boldsymbol{f}$ is differentiable at $\boldsymbol{g}(\boldsymbol{a})$. Is $\boldsymbol{f} \circ \boldsymbol{g} \colon U \to \mathbf{R}^m$ differentiable at $\boldsymbol{a}$? If so, what is its derivative at $\boldsymbol{a}$?

Solution: The composite $\boldsymbol{f} \circ \boldsymbol{g} \colon U \to \mathbf{R}^m$ is differentiable at $\boldsymbol{a}$ and its derivative is $\boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{a}))\boldsymbol{g}'(\boldsymbol{a})$. (You were not asked for a proof, but this is the chain rule and can be found in Theorem 2.1.7 of Lecture 6.) $\square$

(b) (3 marks) Suppose S is a subset of $\mathbf{R}^n$ given by the equation $f(x_1, \dots, x_n) = c$ where $f \colon \mathbf{R}^n \to \mathbf{R}$ is $\mathscr{C}^1$ and c is a constant. Let $\boldsymbol{a} = (a_1, \dots, a_n) \in S$ and assume that $\boldsymbol{\nabla} f$ is non-zero at $\boldsymbol{a}$. Show that $\boldsymbol{\nabla} f(\boldsymbol{a})$ is *normal* to S at $\boldsymbol{a}$, i.e. it is non-zero and if $\boldsymbol{\gamma} \colon I \to \mathbf{R}^n$ is a $\mathscr{C}^1$ map from an open interval I in $\mathbf{R}$, with $\boldsymbol{\gamma}(t) \in S$ for every $t \in I$, and $\theta \in I$ is any point with $\boldsymbol{\gamma}(\theta) = \boldsymbol{a}$, then $\boldsymbol{\nabla} f(\boldsymbol{a})$ is orthogonal to the velocity vector $\boldsymbol{v}_{\boldsymbol{\gamma}}(\theta) := \frac{d\boldsymbol{\gamma}}{dt}\big|_{t=\theta}$.

Solution: This is the same as Problem 2 of HW 4, and the solution can be found at `https://www.cmi.ac.in/~pramath/ANA2/HW/HW4_soln.pdf`. $\square$

(c) (5 marks) Consider the paraboloid P in $\mathbf{R}^3$ given by the equation $ax^2+by^2 = 2z$, and assume the paraboloid is *non-degenerate*, i.e. $ab \neq 0$. Let $(\alpha, \beta, \gamma) \in P$. Find the equation of the tangent plane to P at (α, β, γ).

Solution: Recall from a theorem proved in class (see Theorem 2.2.1 Lecture 8) that a map $\boldsymbol{g}\colon U \to \mathbf{R}^m$, U an open subset of $\mathbf{R}^n$, is $\mathscr{C}^1$ if and only if all its partial derivatives exist and are continuous. Let $f\colon \mathbf{R}^3 \to \mathbf{R}$ be given by

$$f(x,y,z) = ax^2 + by^2 - 2z \qquad ((x,y,z) \in \mathbf{R}^3).$$

Since f is a polynomial in three variables, all its partial derivatives exist and are continuous (since the partial derivatives are also polynomials). Thus $f \in \mathscr{C}^1(\mathbf{R}^3)$. Moreover, $f'(\alpha,\beta,\gamma)^t = \boldsymbol{\nabla} f(\alpha,\beta,\gamma) = (2a\alpha, 2b\beta, -2)$, whence $\boldsymbol{f}'(\alpha,\beta,\gamma) \neq 0$, i.e. it is of rank 1. According to the definition given in the course (see HW 4) the tangent space to P at (α,β,γ) can be defined in this case. Moreover, from the definition of tangent space given in the course, (x,y,z) lies in the tangent plane to P at (α,β,γ), if and only if $(x-\alpha, y-\beta, z-\gamma)$ lies in the null space of $f'(\alpha,\beta,\gamma)$, i.e. if and only if

$$2a\alpha(x-\alpha) + 2b\beta(y-\beta) - 2(z-\gamma) = 0.$$

Simplifying, one gets $a\alpha x + b\beta y - z = a\alpha^2 + b\beta^2 - \gamma = 2\gamma - \gamma = \gamma$. Thus the equation of the tangent plane to P at (α,β,γ) is

$$z = a\alpha x + b\beta y - \gamma.$$

This gives the equation. (If you wish to refresh yourself on all this, look up Homework 4 as well as its Solutions.)

Another way is to note that P is the graph of $\boldsymbol{g}\colon \mathbf{R}^2 \to \mathbf{R}$ where $\boldsymbol{g}(x,y) = \frac{1}{2}(ax^2+by^2)$. This is a $\mathscr{C}^1$ function (the proof being same as the one given above for $\boldsymbol{f}$) whose derivative (written as a matrix) is $\begin{bmatrix} \alpha x & \beta y \end{bmatrix}$. At (α,β) the derivative is $A = \begin{bmatrix} a\alpha & b\beta \end{bmatrix}$. Now apply the formula proven in Problem 4 of Homework 4 to get the same equation as above. □

(6) Let $U = \{(x,y,z) \in \mathbf{R}^3 \mid x > 0 \quad \text{and} \quad y > 0\}$. Let $\boldsymbol{f}\colon U \to \mathbf{R}^2$ be the map given by the formula

$$\boldsymbol{f}(x,y,z) = (-\frac{y}{x}, \frac{1}{2}xy - \arctan z).$$

(a) (4 marks) Prove that $\boldsymbol{f}$ is $\mathscr{C}^1$ and write down $J\boldsymbol{f}(x,y,z)$, the Jacobian matrix form of $\boldsymbol{f}'$ at $(x,y,z) \in U$.

Solution: We have $D_1\boldsymbol{f}(x,y,z) = (\frac{y}{x^2}, \frac{1}{2}y)$, $D_2\boldsymbol{f}(x,y,z) = (-\frac{1}{x}, \frac{1}{2}x)$, and $D_3\boldsymbol{f}(x,y,z) = (0, -\frac{1}{1+z^2})$. Clearly all three partial derivatives are continuous on U, since x is not zero on U and $1+z^2$ cannot vanish anywhere. It follows, from a theorem proved in class that $\boldsymbol{f}$ is $\mathscr{C}^1$. Since the i^{th} column of $J\boldsymbol{f}$ is $D_i\mathbf{f}$ we get

$$(J\boldsymbol{f})(x,y,z) = \begin{bmatrix} \frac{y}{x^2} & -\frac{1}{x} & 0 \\ \\ \frac{1}{2}y & \frac{1}{2}x & -\frac{1}{1+z^2}. \end{bmatrix}$$

□

(b) (6 marks) Let $(x_\circ, y_\circ, z_\circ)$ be a point in U with $\boldsymbol{f}(x_\circ, y_\circ, z_\circ) = (a, b)$. Show that there is an open interval I in $\mathbf{R}$ containing $x_\circ$ and a $\mathscr{C}^1$ map $(\gamma_1, \gamma_2)\colon I \to (0, \infty) \times \mathbf{R}$ such that $\gamma_1(x_\circ) = y_\circ$, $\gamma_2(x_\circ) = z_\circ$, $(t, \gamma_1(t), \gamma_2(t)) \in U$ for $t \in I$, and $\boldsymbol{f}(t, \gamma_1(t), \gamma_2(t)) = (a, b)$ for all $t \in I$.

Solution: Since $-\frac{1}{1+z^2}$ and $-\frac{1}{x}$ never vanish on U, the submatrix of the matrix $(J\boldsymbol{f})(x, y, z)$ above given by the last two columns is invertible for every $(x, y, z) \in U$ (the determinant of the submatrix is $\frac{1}{x(1+z^2)}$ which is non-zero on U). Since $(x_\circ, y_\circ, z_\circ) \in U$, the observation applies to this point too. The implicit function theorem gives the rest. (You should state the theorem, and deduce the existence of γ_1 and γ_2 with the required properties from the statement for full marks.) ☐

(7) (a) (4 marks) State the Inverse Function Theorem.

Solution: This is just Theorem 1.2.1 of Lecture 10. ☐

(b) (6 marks) Let a be a positive real number and $\boldsymbol{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$ the map given by the formula $\boldsymbol{f}(x, y) = (x, (x^2+y^2)^2 - 2a(x^2-y^2))$ for $(x, y) \in \mathbf{R}^2$. Let C be the curve in $\mathbf{R}^2$ defined by the equation $(x^2+y^2)^2 = 2a(x^2-y^2)$. Find all points (α, β) in C for which the hypotheses of the inverse function theorem fail for $\boldsymbol{f}$.

Solution: Since all components of $\boldsymbol{f}$ are polynomials, all it partial derivatives exist and are continuous. Thus $\boldsymbol{f}$ is $\mathscr{C}^1$. Clearly

$$(J\boldsymbol{f})(x, y, z) = \begin{bmatrix} 1 & 0 \\ 4x(x^2+y^2) - 4ax & 4y(x^2+y^2) + 4ay \end{bmatrix}.$$

The determinant of this matrix is

$$4y(x^2+y^2) + 4ay = 4y(x^2+y^2+a)$$

and hence $\boldsymbol{f}'(x, y)$ is non-invertible if and only if $4y(x^2+y^2+a) = 0$. Since $a > 0$, this can only happen when $y = 0$. For the hypotheses of the inverse function theorem to fail on a point (α, β) on C, we require therefore that $\beta = 0$ and that (α, β) satisfy the equation $(x^2+y^2)^2 = 2a(x^2-y^2)$. This means $\alpha^4 = 2a\alpha^2$, i.e. $\alpha^2(\alpha^2 - 2a) = 0$. The solutions are $\alpha = 0, \pm\sqrt{2a}$. Thus the points $(\alpha, \beta) \in C$ which we were required to find are $(-\sqrt{2a}, 0)$, $(0, 0)$, and $(\sqrt{2a}, 0)$. ☐