

Equivalence of norms on finite dimensional Vector Space

Analysis - II (Tutorial)

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If we are given two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on some finite-dimensional vector space V over \mathbb{C} , a very useful fact is that they are always within a constant factor of one another. Specifically, there exists a pair of real numbers $0 < C_1 \leq C_2$ such that, for all $x \in V$, the following inequality holds:

$$C_1\|x\|_b \leq \|x\|_a \leq C_2\|x\|_b.$$

Note that any *finite*-dimensional vector space, by definition, is spanned by a *basis* e_1, e_2, \dots, e_n where n is the *dimension of the vector space*.

Any vector x can be written as

$$x = \sum_{i=1}^n \alpha_i e_i$$

where the α_i are some scalars depending on x .

Step 1: It is sufficient to consider $\|\cdot\|_b = \|\cdot\|_1$ (transitivity).

First, us define an L_1 -style norm by

$$\|x\|_1 = \sum_{i=1}^n |\alpha_i|.$$

(It is easy to see this is a norm. The linear independence of any basis $\{e_i\}$ means that $x \neq 0 \iff \alpha_j \neq 0$ for some $j \iff \|x\|_1 > 0$. The triangle inequality and the scaling property are obvious and follow from the usual properties of L_1 norms on $\alpha \in \mathbb{C}^n$.)

We will show that it is sufficient for to prove that $\|\cdot\|_a$ is equivalent to $\|\cdot\|_1$, because norm equivalence is *transitive*: if two norms are equivalent to $\|\cdot\|_1$, then they are equivalent to each other. In particular, suppose both $\|\cdot\|_a$ and $\|\cdot\|_{a'}$ are equivalent to $\|\cdot\|_1$ for constants $0 < C_1 \leq C_2$ and $0 < C'_1 \leq C'_2$, respectively:

$$\begin{aligned} C_1\|x\|_1 &\leq \|x\|_a \leq C_2\|x\|_1, \\ C'_1\|x\|_1 &\leq \|x\|_{a'} \leq C'_2\|x\|_1. \end{aligned}$$

It immediately follows that

$$\frac{C'_1}{C_2}\|x\|_a \leq \|x\|_{a'} \leq \frac{C'_2}{C_1}\|x\|_a,$$

and hence $\|\cdot\|_a$ and $\|\cdot\|_{a'}$ are equivalent.

Step 2: It is sufficient to consider only x with $\|x\|_1 = 1$

We wish to show that

$$C_1\|x\|_1 \leq \|x\|_a \leq C_2\|x\|_1,$$

is true for all $x \in V$ for some C_1, C_2 . It is trivially true for $x = 0$, so we need only consider $x \neq 0$, in which case we can divide by $\|x\|_1$ to obtain the condition

$$C_1 \leq \|u\|_a \leq C_2,$$

where $u = x/\|x\|_1$ has norm $\|u\|_1 = 1$.

Step 3: Any norm $\|\cdot\|_a$ is continuous under $\|\cdot\|_1$

We wish to show that any norm $\|\cdot\|_a$ is a continuous function on V under the topology induced by the norm $\|\cdot\|_1$. That is, we wish to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x - x'\|_1 < \delta \implies \left| \|x\|_a - \|x'\|_a \right| < \epsilon.$$

We prove this in two steps. First, by the triangle inequality on $\|\cdot\|_a$, it follows that

$$\begin{aligned} \|x\|_a - \|x'\|_a &= \|x' + (x - x')\|_a - \|x'\|_a \leq \|x - x'\|_a \\ \|x'\|_a - \|x\|_a &= \|x - (x - x')\|_a - \|x\|_a \leq \|x - x'\|_a \end{aligned}$$

and hence

$$\left| \|x\|_a - \|x'\|_a \right| \leq \|x - x'\|_a$$

Second, applying the triangle inequality again, and writing $x = \sum_{i=1}^n \alpha_i e_i$ and $x' = \sum_{i=1}^n \alpha'_i e_i$ in our basis, we obtain

$$\|x - x'\|_a \leq \sum_{i=1}^n |\alpha_i - \alpha'_i| \cdot \|e_i\|_a \leq \|x - x'\|_1 \left(\max_i \|e_i\|_a \right).$$

Therefore, if we choose

$$\delta = \frac{\epsilon}{\max_i \|e_i\|_a},$$

it immediately follows that

$$\|x - x'\|_1 < \delta \implies \left| \|x\|_a - \|x'\|_a \right| \leq \|x - x'\|_a < \epsilon.$$

Step 4: The maximum and minimum of $\|\cdot\|_a$ on the unit sphere

It is a standard theorem of analysis, the *extreme value theorem*, that a continuous function (e.g. $\|\cdot\|_a$, from step 3) on compact set (e.g. the unit “sphere” defined by $\{u \text{ for } \|u\|_1 = 1\}$, a closed and bounded set) must achieve a maximum and minimum value on the set. Let

$$C_1 = \min_{\|u\|_1=1} \|u\|_a,$$

$$C_2 = \max_{\|u\|_1=1} \|u\|_a.$$

Since $u \neq 0$ for $\|u\|_1 = 1$, it follows that $C_2 \geq C_1 > 0$ and

$$C_1 \leq \|u\|_a \leq C_2$$

as required by step 2.