

LECTURE 4

Date of Lecture: January 22, 2020

1. Compactness

1.1. In Analysis-I you were taught the Heine-Borel theorem on \mathbf{R} , namely that a subset K of \mathbf{R} is compact if and only if it is closed and bounded. This turns out to be true in \mathbf{R}^n as we will see in the next lecture. But first let us define compact metric spaces.

We point out that if (X, d) is a metric space, S a non-empty subset of X , then the restriction of d to $S \times S$ is a metric on S , making $(S, d|_{S \times S})$ into a metric space. As we have remarked earlier, a subset D of S is open (respectively closed) in the metric space $(S, d|_{S \times S})$ if and only if D is open (respectively closed) in S , i.e. if and only if there exists an open (respectively closed) set W in X such that $D = S \cap W$.

Definition 1.1.1. Let (X, d) be a metric space. A subset K of X is said to be *compact* if every open cover of K in X has a finite subcover. X is said to be a *compact metric space* if it is a compact subset of itself.

Lemma 1.1.2. Let (X, d) be a metric space, K a non-empty subset of X , and d_K the restriction of d to $K \times K$. Then K is a compact subset of X if and only if (K, d_K) is a compact metric space.

Proof. Suppose K is a compact subset of X . Let \mathcal{V} be a cover of K by sets which are open in K , i.e. by sets which are open with respect to (K, d_K) . Then for each $V \in \mathcal{V}$ we can find an open set U_V in X such that $V = K \cap U_V$. Clearly $\mathcal{U} := \{U_V \mid V \in \mathcal{V}\}$ is an open cover of K in X . Since K is compact in X , there exist $V_1, \dots, V_n \in \mathcal{V}$ such that $\{U_{V_1}, \dots, U_{V_n}\}$ is an open cover of K . Since $V_i = U_{V_i} \cap K$, clearly $\{V_1, \dots, V_n\}$ is a cover of K , and this gives us a finite subcover of \mathcal{V} with respect to (K, d_K) . Thus (K, d_K) is a compact metric space.

Conversely, suppose (K, d_K) is a compact metric space, and \mathcal{U} is an open cover of K in X . Let $\mathcal{V} = \{U \cap K \mid U \in \mathcal{U}\}$. Then \mathcal{V} is a cover of K by sets which are open with respect to (K, d_K) . Since (K, d_K) is a compact metric space, \mathcal{V} has a finite subcover $\{V_1, \dots, V_n\}$. Now each $V_j = U_j \cap K$ for some $U_j \in \mathcal{U}$, by our definition of \mathcal{V} . It follows that $\{U_1, \dots, U_n\}$ is a finite subcover of the cover \mathcal{U} of K . Hence K is a compact subset of X . \square

Remark 1.1.3. The philosophical content of the lemma above is that compactness is an *intrinsic* property of a subset of a metric space. A subset is compact in a larger metric space if and only if it is compact in its own right. In particular, if a compact metric space K is embedded in two different metric spaces as a submetric space, then it is a compact subset of both those metric spaces. Thus the embedding of $[0, 1]$ with the standard metric into \mathbf{R}^2 by the parametrization $t \mapsto (\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})$ gives us a compact subset of \mathbf{R}^2 . As does the standard embedding $[0, 1] \subset \mathbf{R}$.

Proposition 1.1.4. A closed subset C of a compact metric space X is compact.

Proof. Let \mathcal{V} be an open cover of C in X . Let $U = X \setminus C$ and let $\mathcal{U} = \mathcal{V} \cup \{U\}$. Then \mathcal{U} is an open cover of X . Since X is compact there is a finite subset \mathcal{U}' of \mathcal{U} which covers X . Clearly $\mathcal{V}' = \mathcal{U}' \setminus \{U\}$ is a finite subset of \mathcal{V} which covers C . Thus C is compact. \square

Definition 1.1.5. A subset S of a metric space X is said to be bounded if there exists $x \in X$ and a positive real number r such that $S \subset B(x, r)$.

Remark 1.1.6. If $(X, \|\cdot\|)$ is a normed linear space over \mathbf{K} , where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$, and $\|\cdot\|'$ is a norm on X such that $\|\cdot\| \sim \|\cdot\|'$, then it is easy to see a set S is bounded with respect to the metric induced by $\|\cdot\|$ if and only if it is with respect to the metric induced by $\|\cdot\|'$. In particular, by Remark 1.1.6 in [Lecture 3](#), on \mathbf{K}^n , bounded sets with respect to $\|\cdot\|_\infty$ are exactly the bounded sets with respect to $\|\cdot\|_2$.

Proposition 1.1.7. A compact subset of a metric space is closed and bounded.

Proof. Let (X, d) be a metric space and K a compact subset of X . Pick any $x \in X$. Then $\mathcal{U} = \{B(x, n) \mid n \in \mathbf{N}\}$ is an open cover of X and hence of K . There is a finite subset $\{B(x, n_1), \dots, B(x, n_k)\}$ of \mathcal{U} which covers K . Without loss of generality we may assume $n_1 < \dots < n_k$. It follows that $K \subset B(x, n_k)$. Hence K is bounded.

It remains to show that K is closed. Let $U = X \setminus K$. We have to show that U is open in X . Let $x \in U$ and $z \in K$. Let $\delta_z = d(x, z)/3$ and set

$$(*) \quad G_z = B(x, \delta_z), \quad \text{and} \quad V_z = B(z, \delta_z).$$

We claim that $G_z \cap V_z = \emptyset$. To see this, suppose $y \in G_z \cap V_z$. Then

$$d(x, z) \leq d(x, y) + d(y, z) \leq \delta_z + \delta_z = \frac{2}{3}d(x, z).$$

Now x cannot equal z since they lie in complementary sets. Hence $d(x, z) > 0$, and therefore the inequality $d(x, z) \leq \frac{2}{3}d(x, z)$ is impossible. Hence $G_z \cap V_z = \emptyset$.

Let $\mathcal{V} = \{V_z \mid z \in K\}$. This is an open cover of K . Since K is compact, there exist $z_1, \dots, z_n \in K$ such that

$$K \subset \bigcup_{i=1}^n V_{z_i}.$$

Let $U_x = \bigcap_{i=1}^n G_{z_i}$. Clearly $U_x \cap V_{z_i} = \emptyset$ for $1 \leq i \leq n$, and hence $U_x \cap K = \emptyset$. Thus $U_x \subset U$. From (*), U_x is an open neighbourhood of x since each G_{z_i} is. It follows that $U = \bigcup_{x \in U} U_x$. Thus U is open. \square

2. Towards Heine-Borel in \mathbf{R}^n

2.1. Boxes and cubes. We begin with some definitions.

Definition 2.1.1. A *box* in \mathbf{R}^n is a set B of the form

$$B = I_1 \times \dots \times I_n$$

where each I_j occurring as a factor in the above product is an interval in \mathbf{R} . (These intervals could be open, closed, half open, half closed, independently of each other.) A box $B = I_1 \times \dots \times I_n$ is said to be an *open box* if each of the I_j are open. It is called a *closed box* if each of them is a closed interval. It is called a *cube* if each interval I_j is a bounded interval and all of them have the same length. An *open cube* is a cube which is an open box. A *closed cube* is a cube which is a closed box.

Remark 2.1.2. Open cubes are exactly the same as open balls in $(\mathbf{R}^n, \|\cdot\|_\infty)$. The centre of an open cube coincides with the centre in its representation as an open ball in the $\|\cdot\|_\infty$ norm. A box is bounded (in $\|\cdot\|_\infty$ or in $\|\cdot\|_2$) if and only if every interval occurring in its product decomposition is bounded. In particular a set $S \subset \mathbf{R}^n$ is bounded if and only if it is a subset of a bounded box. In fact a little thought shows that S is bounded if and only if it is a subset of a closed bounded box since every bounded interval is a subset of a closed bounded interval. Finally, it is clear from Corollary 2.2.8 of [Lecture 3](#) that a box is open (respectively closed) if and only if each interval occurring as a factor in its product decomposition is open (respectively closed). In particular “an open box = a box that is open in $(\mathbf{R}^n, \|\cdot\|_\infty)$ ”. The analogous statement for closed box also follows from the above observations.

Proposition 2.1.3. *Let $K = I_1 \times \cdots \times I_n$ be a closed bounded box in \mathbf{R}^n . Let $\mathcal{Q} = \{Q_\lambda \mid \lambda \in \Lambda\}$ be an open cover of K with respect to $\|\cdot\|_\infty$ with each Q_λ an open cube. Then \mathcal{Q} has a finite subcover of K .*

Proof. For any cube $Q = J_1 \times \cdots \times J_n$, let

$$(\dagger) \quad B_Q = J_1 = \pi_1(Q) \quad \text{and} \quad C_Q = J_2 \times \cdots \times J_n \subset \mathbf{R}^{n-1}.$$

Note that B_Q is an interval in \mathbf{R} and C_Q a cube in \mathbf{R}^{n-1} . If Q is an open cube, then B_Q is an open interval in \mathbf{R} and C_Q is an open cube in \mathbf{R}^{n-1} .

We will prove the proposition by induction on n . If $n = 1$, this is the classical Heine-Borel theorem on \mathbf{R} . Assume $n > 1$ and that the theorem is true for $n - 1$. Set $K' = I_2 \times \cdots \times I_n \subset \mathbf{R}^{n-1}$.

Let $t \in I_1$. Let \mathcal{Q}_t be the subcollection of \mathcal{Q} which has a non-trivial intersection with $\{t\} \times K'$. Note that

$$\mathcal{Q}_t = \{Q \in \mathcal{Q} \mid t \in B_Q\}.$$

where B_Q is as in (\dagger) . Now $\{C_Q \mid Q \in \mathcal{Q}_t\}$ is a cover of K' by open cubes, where C_Q is as in (\dagger) . By our induction hypothesis, there exists a finite subset \mathcal{F}_t of \mathcal{Q}_t such that $\{C_Q \mid Q \in \mathcal{F}_t\}$ is a cover of K' . It follows that \mathcal{F}_t is a cover of $\{t\} \times K'$. Since \mathcal{F}_t is finite, the collection $\{B_Q \mid Q \in \mathcal{F}_t\}$ is a finite collection of intervals in \mathbf{R} and every interval in this collection contains t . Thus

$$U_t := \bigcap_{Q \in \mathcal{F}_t} B_Q$$

is an open interval in \mathbf{R} containing t . Moreover,

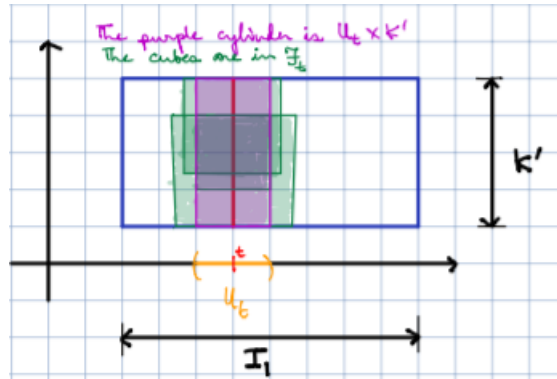
$$(\ddagger) \quad U_t \times K' \subset U_t \times \bigcup_{Q \in \mathcal{F}_t} C_Q = \bigcup_{Q \in \mathcal{F}_t} (U_t \times C_Q) \subset \bigcup_{Q \in \mathcal{F}_t} (B_Q \times C_Q) = \bigcup_{Q \in \mathcal{F}_t} Q.$$

(See picture at the end of the proof.)

Clearly $\mathcal{U} = \{U_t \mid t \in I_1\}$ is an open cover of I_1 . By the Heine-Borel theorem on \mathbf{R} , there exist $t_1, \dots, t_m \in I_1$ such that $\{U_{t_j} \mid 1 \leq j \leq m\}$ is an open cover of I_1 . Now

$$K = I_1 \times K' \subset \left(\bigcup_{i=1}^m U_{t_i} \right) \times K' = \bigcup_{i=1}^m (U_{t_i} \times K') \subset \bigcup_{i=1}^m \bigcup_{Q \in \mathcal{F}_{t_i}} Q,$$

where we are using (\ddagger) for the last inclusion in the above chain of inclusions. Thus $\{Q \in \mathcal{F}_{t_i} \mid 1 \leq i \leq m\}$ is a finite subcover of \mathcal{Q} covering K .



□

About these notes. These course notes are a reasonably faithful record of the lectures given at the [Chennai Mathematical Institute](https://www.cmi.ac.in/) (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit <https://www.cmi.ac.in/~pramath/teaching.html#ANA2>. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.