LECTURE 3

Date of Lecture: January 20, 2020

1. Norms on $\mathbf{K}^n$

We will see later in the course that on a finite dimensional space X over $\mathbf{K}$, all norms give the same collection of open sets and the same collection of closed sets and convergence of a sequence in one norm implies its convergence in all other norms on X. The proof of that will have to wait until we do the Heine-Borel theorem on $\mathbf{K}^n$. However, we can do a special case right now.

1.1. The spaces $(\mathbf{K}^n, \|\ \|_\infty)$ and $(\mathbf{K}^n, \|\ \|_2)$.

Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{K}^n$. Then

$$|x_i| \le \left(|x_1|^2 + \cdots + |x_n|\right)^{\frac{1}{2}} = \|\boldsymbol{x}\|_2$$

for $i = 1, \dots, n$. It follows that $\max_{i=1,\dots,n} |x_i| \le \|\boldsymbol{x}\|_2$. In other words

$$\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_2. \tag{1.1.1}$$

One can show easily that this proves that if $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_2)$ then $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_\infty)$. The proof is given in a more general situation in the proof Lemma 1.1.5 below.

On the other hand, $|x_i|^2 \le \|\boldsymbol{x}\|_\infty^2$ for every $i = 1, \dots, n$, whence on summing over i we get $\sum_{i=1}^n |x_i|^2 \le n\|\boldsymbol{x}\|_\infty^2$. In other words

$$\|\boldsymbol{x}\|_2 \le \sqrt{n}\|\boldsymbol{x}\|_\infty. \tag{1.1.2}$$

This inequality shows that if $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_\infty)$, then $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_2)$. (See proof of Lemma 1.1.5 below.)

Definition 1.1.3. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a $\mathbf{K}$-vector space X are said to be *equivalent* if there exist constants $c, C > 0$ such that

$$c\|\boldsymbol{x}\|' \le \|\boldsymbol{x}\| \le C\|\boldsymbol{x}\|' \qquad (\boldsymbol{x} \in X).$$

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms, we write $\|\cdot\| \sim \|\cdot\|'$.

It is clear that $\sim$ is an equivalence relation on the set of norms on X.

Example 1.1.4. From (1.1.1) and (1.1.2) we see that on $\mathbf{K}^n$, $\|\ \|_\infty \sim \|\ \|_2$. More generally, we will show later in this course that if $\|\ \|$ and $\|\ \|'$ are two norms on $\mathbf{K}^n$ then they are equivalent.

Lemma 1.1.5. *Let $\|\ \|$ and $\|\ \|'$ be equivalent norms on the $\mathbf{K}$-vector space X, $\{\boldsymbol{x}_m\}$ a sequence in X, and S a subset of X. Then*

(i) *The sequence $\{\boldsymbol{x}_n\}$ converges to $\boldsymbol{x} \in X$ with respect to $\|\ \|$ if and only if it does so with respect to $\|\ \|'$.*
(ii) *S is open (respectively closed) with respect to $\|\ \|$ if and only if it is open (respectively closed) with respect to $\|\ \|'$.*

Proof. Let $c, C > 0$ be such that

$$(*) \qquad c\| \ \|' \leq \| \ \| \leq C\| \ \|'.$$

Suppose $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ in $(X, \| \ \|)$. Given $\epsilon > 0$ we have $N \in \mathbf{N}$ such that $\|\boldsymbol{x}_n - \boldsymbol{x}\| < c\epsilon$ for $n \geq N$. This means $c\|\boldsymbol{x}_n - \boldsymbol{x}\|' \leq \|\boldsymbol{x}_n - \boldsymbol{x}\| < c\epsilon$ for $n \geq N$, i.e. $\|\boldsymbol{x}_n - \boldsymbol{x}\|' < \epsilon$ for $n \geq N$, showing that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ in $(X, \| \ \|')$. The converse follows because $\sim$ is an equivalence relation. This proves (i).

To prove part (ii), it is enough to prove that S is open in with respect to $\| \ \|$ if and only if it is so with respect to $\| \ \|'$, for the case where S is closed then follows by taking complements. To prove this, we need notations. Let $B(\boldsymbol{x}, r)$ and $B'(\boldsymbol{x}, r)$ denote the open balls of radius r centred at $\boldsymbol{x}$ with respect to $\| \ \|$ and $\| \ \|'$ respectively. In other words $B(\boldsymbol{x}, r) = \{\boldsymbol{z} \in X \mid \|\boldsymbol{z} - \boldsymbol{x}\| < r\}$ and $B'(\boldsymbol{x}, r) = \{\boldsymbol{z} \in X \mid \|\boldsymbol{z} - \boldsymbol{x}\|' < r\}$. From $(*)$ we get

- $\|\boldsymbol{y} - \boldsymbol{x}\|' < r \Rightarrow \|\boldsymbol{x} - \boldsymbol{y}\| < Cr$, and
- $\|\boldsymbol{y} - \boldsymbol{x}\| < r \Rightarrow \|\boldsymbol{x} - \boldsymbol{y}\|' < r/c$,

whence,

$$(\dagger) \qquad B'(\boldsymbol{x}, r) \subset B(\boldsymbol{x}, Cr), \quad \text{and} \quad B(\boldsymbol{x}, r) \subset B'(\boldsymbol{x}, r/c).$$

Let S be open with respect to $\| \ \|$. According to Lemma 1.1.7 of Lecture 2, for each $\boldsymbol{x} \in S$ there is an open ball $B(\boldsymbol{x}, r_{\boldsymbol{x}}) \subset S$. By $(\dagger)$ we have $B'(\boldsymbol{x}, r_{\boldsymbol{x}}/C) \subset S$. It follows that $S = \cup_{\boldsymbol{x}\in S} B'(\boldsymbol{x}, r_{\boldsymbol{x}}/C)$ and hence S is open with respect to $\| \ \|'$. By symmetry, if S is open with respect to $\| \ \|'$ it is so with respect to $\| \ \|$. Part (ii) follows. □

Remark 1.1.6. From (1.1.1) and (1.1.2) we see that that the open (respectively closed) sets of $(\mathbf{K}^n, \| \ \|_\infty)$ coincide with those of $(\mathbf{K}^n, \| \ \|_2)$. We also see that a sequence $\{\boldsymbol{x}_m\}$ converges to $\boldsymbol{x}$ in $(\mathbf{K}^n, \| \ \|_\infty)$ if and only if it converges to $\boldsymbol{x}$ in $(\mathbf{K}^n, \| \ \|_2)$.

Corollary 1.1.7. *Let $\{\boldsymbol{x}_m\}$ be a sequence in $\mathbf{K}^n$, say $\boldsymbol{x}_m = (x_{1m}, \dots, x_{nm})$. Then*

$$\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$$

with respect to $\| \ \|_2$ if and only if

$$\lim_{m\to\infty} x_{im} = x_i$$

for $i = 1, \dots, n$.

Proof. By Remark 1.1.6 we have, $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect $\| \ \|_2$ if and only if $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect to $\| \ \|_\infty$. Now $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect to $\| \ \|_\infty$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$\max_{i=1,\dots,n} |x_{im} - x_i| < \epsilon \qquad (m \geq N).$$

The above it true if and only if

$$|x_{im} - x_i| < \epsilon \qquad (m \geq N; i = 1, \dots, n).$$

i.e. if and only if $\lim_{m\to\infty} x_{im} = x_i$ for every $i \in \{1, \dots, n\}$. □

2. **Continuity revisited**

2.1. **Inverse images.** Let X and Y be normed linear spaces over $\mathbf{K}$, $S \subset X$, and $f\colon S \to Y$ a map. For $x \in X$ and $r > 0$, let $B_X(x, r)$ denote the open ball in X of radius r icentred at x, and similarly, for $y \in Y$, let $B_Y(y, r)$ denote the open ball in Y of radius r centred at y. Fix $a \in X$ and let $b = f(a)$. Suppose f is continuous at a. The usual $\epsilon - \delta$ definition of continuity of f at a can be rephrased as saying that given $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_X(a, \delta) \cap S) \subset B_Y(b, \epsilon)$. If U is a neighbourhood of b, then $U \supset B_Y(b, \epsilon)$ for some $\epsilon > 0$ and hence we can find an open neighbourhood V of a (take $V = B_X(a, \delta)$ for suitable $\delta > 0$) such that $f(V \cap S) \subset U$. Conversely, if for every open neighbourhood U of b, we can find an open neighbourhood V of a such that $f(V \cap S) \subset U$, then clearly, for every $\epsilon > 0$ we can find $\delta > 0$ such that $f(B_X(a, \delta) \cap S) \subset B_Y(b, \epsilon)$, and hence f is continuous at a.

Proposition 2.1.1. *Let X, Y be normed linear spaces over $\mathbf{K}$, S a subset of X and $f\colon S \to Y$ a map. Then f is continuous on S if and only if $f^{-1}(U)$ is open in S for every open set U in Y.*

Proof. If f is continuous and U is open in Y, then U is a neighbourhood of every point in $f(S)$. From the discussion above the statement of the proposition, if $a \in f^{-1}(U)$, then there is an open neighbourhood V_a of a such that $f(V_a \cap S)$ is a subset of U. Let $V = \cup_{a \in f^{-1}(U)} V_a$. Then $f^{-1}(U) \subset V$ and $f(V \cap S) \subset U$. Thus $f^{-1}(U) = V \cap S$, and hence $f^{-1}(U)$ is open in S.

Conversely, suppose $f^{-1}(U)$ is open in S for every open set U in Y. Let $a \in X$ and let $b = f(a)$. If U is an open neighbourhood of b, then $f^{-1}(U)$ is open in S and hence there exists an open set V in X such that $V \cap S = f^{-1}(U)$. Now V is an open neighbourhood of a such that $f(V \cap S) \subset U$ and hence from the discussion above the statement of the proposition, f is continuous at a. Since $a \in X$ is arbitrary, we are done. □

An immediate corollary is:

Corollary 2.1.2. *The map f is continuous on S if and only if $f^{-1}(F)$ is closed in S for every closed set F in Y.*

Remark 2.1.3. The notion of a metric space, and of continuity of maps between metric spaces was defined in HW 2. One can define open balls $B_X(x, r)$ on a metric space (X, d) for $x \in X$ and $r > 0$ just as we did for normed spaces, and so define open sets. The basic theorems continue to hold (e.g. the intersection of two open sets is open). The above proof works *mutatis mutandis* in that situation too and we have that a map $f\colon X \to Y$ between metric spaces is continuous if and only if $f^{-1}(U)$ is open in X for every open set U in Y.

2.2. **Product sets.** Suppose r and d are non-negative integers such that $r + d = n$. Then we can regard $\mathbf{K}^n$ as $\mathbf{K}^r \times \mathbf{K}^d$ and write elements $\boldsymbol{x} \in \mathbf{K}^n$ uniquely as $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ with $\boldsymbol{x}_1 \in \mathbf{K}^r$ and $\boldsymbol{x}_2 \in \mathbf{K}^d$. One checks easily that

$$\|\boldsymbol{x}\|_2^2 = \|\boldsymbol{x}_1\|_2^2 + \|\boldsymbol{x}_2\|_2^2, \quad \text{and} \quad \|\boldsymbol{x}\|_\infty = \max(\|\boldsymbol{x}_1\|_\infty, \|\boldsymbol{x}_2\|_\infty). \tag{2.2.1}$$

Recall that $\|\ \|_2$ and $\|\ \|_\infty$ are equivalent. We will show that if $A \subset \mathbf{K}^r$ and $B \subset K^d$ then $A \times B$ is open (respectively closed) in $(\mathbf{K}^n, \|\ \|_2)$ if and only if A and B are open (respectively closed) in $(\mathbf{K}^r, \|\ \|_2)$ and $(\mathbf{K}^d, \|\ \|_2)$ respectively.

We begin with the following lemma.

Lemma 2.2.2. *For $i = 1, \dots, n$ let*

$$\pi_i \colon \mathbf{K}^n \longrightarrow \mathbf{K} \qquad (i = 1, \dots, n)$$

be the i^{th} projection map, i.e. the map $(x_1, \dots, x_n) \mapsto x_i$. Then π_i is continuous on $(\mathbf{K}^n, \|\ \|_2)$ for each $i \in \{1, \dots, n\}$.

Proof. Clearly $(\pi_1, \dots, \pi_n) \colon \mathbf{K}^n \to \mathbf{K}^n$ is the identity map on $\mathbf{K}^n$ which is continuous. By Problem 3 of HW 2 we are done. ☐

Corollary 2.2.3. *Let J be a non-empty subset of $\{1, \dots, , n\}$, say $J = \{j_1, \dots, j_d\}$ with $j_1 < \dots < j_d$. Let*

$$\pi_J \colon (\mathbf{K}^n, \|\ \|_2) \longrightarrow (\mathbf{K}^d, \|\ \|_2)$$

be the projection map $(x_1, \dots, x_n) \mapsto (x_{j_1}, \dots, x_{j_d})$. Then π_J is continuous.

Proof. Clearly $\pi_J = (\pi_{j_1}, \dots, \pi_{j_d})$, and therefore by Lemma 2.2.2 and Problem 3 of HW 2, we are done ☐

2.2.4. Suppose J, π_J are as in Corollary 2.2.3. Let $r = n - d$ and let I be the complement of J in $\{1, \dots, n\}$. Then, for any $\boldsymbol{z} \in \mathbf{K}^d$ we have a natural identification of $\pi_J^{-1}(\boldsymbol{z})$ with $\mathbf{K}^r$. In greater detail, if $I = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$, then we have a natural permutation σ on $\{1, \dots, n\}$, namely the permutation

$$\sigma(k) = \begin{cases} i_k & \text{if } 1 \le k \le r \\ j_{k-r} & \text{if } r + 1 \le k \le n. \end{cases}$$

Consider the map

$$\begin{aligned} \Phi_\sigma \colon \mathbf{K}^n &\longrightarrow \mathbf{K}^n \\ (x_1, \dots, x_n) &\longmapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \end{aligned}$$

Then Φ_σ is a vector-space isomorphism which preserves both norms of current interest, namely $\|\ \|_2$ and $\|\ \|_\infty$, i.e. $\|\Phi_\sigma(\boldsymbol{x})\|_2 = \|\boldsymbol{x}\|_2$ and $\|\Phi_J(\boldsymbol{x})\|_\infty = \|\boldsymbol{x}\|_\infty$ for $\boldsymbol{x} \in \mathbf{K}^n$. Such norm preserving linear maps are called *isometric isomorphisms*. Identifying $\mathbf{K}^n$ with $\mathbf{K}^r \times \mathbf{K}^d$ we see that

$$\Phi_\sigma(\mathbf{K}^r \times \{\boldsymbol{z}\}) = \pi_J^{-1}(\boldsymbol{z}).$$

We therefore have a bijective map (i.e. a one-to-one and onto map)

$$\varphi_{J,\boldsymbol{z}} \colon \mathbf{K}^r \longrightarrow \pi_J^{-1}(\boldsymbol{z}), \qquad \boldsymbol{y} \longmapsto \Phi_\sigma(\boldsymbol{y}, \boldsymbol{z}), \tag{2.2.4.1}$$

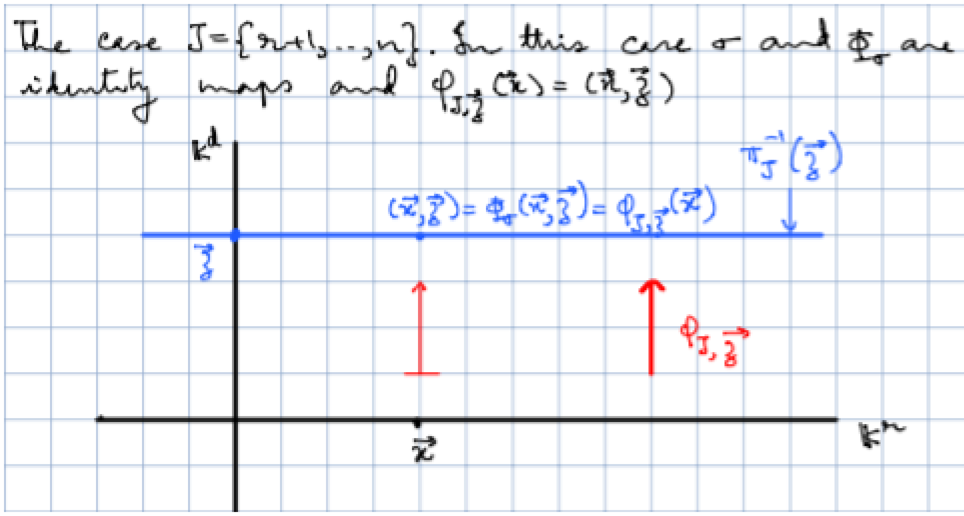
where, of course, we regard $(\boldsymbol{y}, \boldsymbol{z})$ as a point of $\mathbf{K}^n$ via the identification $\mathbf{K}^n = \mathbf{K}^r \times \mathbf{K}^d$. Using the fact that Φ_σ is an isometry for both $\|\ \|_2$ and $\|\ \|_\infty$ we get

$$\begin{aligned} \|\boldsymbol{a} - \boldsymbol{b}\|_2 &= \|\varphi_{J,\boldsymbol{z}}(\boldsymbol{a}) - \varphi_{J,\boldsymbol{z}}(\boldsymbol{b})\|_2\,, \\ \|\boldsymbol{a} - \boldsymbol{b}\|_\infty &= \|\varphi_{J,\boldsymbol{z}}(\boldsymbol{a}) - \varphi_{J,\boldsymbol{z}}(\boldsymbol{b})\|_\infty. \end{aligned} \tag{2.2.4.2}$$

Indeed, using (2.2.1) and the fact that Φ_σ is an isometry, we have

$$\|\boldsymbol{a} - \boldsymbol{b}\|_2 = \|(\boldsymbol{a}, \boldsymbol{z}) - (\boldsymbol{b}, \boldsymbol{z})\|_2 = \|\Phi_\sigma(\boldsymbol{a}, \boldsymbol{z}) - \Phi_\sigma(\boldsymbol{b}, \boldsymbol{z})\|_2 = \|\varphi_{J,\boldsymbol{z}}(\boldsymbol{a}) - \varphi_{J,\boldsymbol{z}}(\boldsymbol{b})\|_2,$$

and one can reproduce an identical calculation for $\|\ \|_\infty$.



The above discussion gives us the following.

Lemma 2.2.5. *In the situation discussed in §§2.2.4, a subset S of $\mathbf{K}^r$ is open (respectively closed) in $\mathbf{K}^r$ if and only if $\varphi_{J,\boldsymbol{z}}(S)$ is open (respectively closed) in $\pi_J^{-1}(\boldsymbol{z})$, where $\mathbf{K}^r$ and $\mathbf{K}^n$ are given the $\| \, \|_\bigstar$ norm for $\bigstar \in \{2, \infty\}$.*

Remark: The proof is best given in terms of metric spaces, but since we haven't covered that too well (there are some homework exercises defining them and getting you to work out their first properties), I am offering a more roundabout official proof below this remark. However, the following observation may be useful in helping you work out a more natural proof. If (X, d) is a metric space and $\varnothing \neq Y \subset X$, then Y is a metric space too, with the metric inherited from X. If we write $B_Y(y, r)$ and $B_X(x, r)$ for balls of radius r around points $y \in Y$ and $x \in X$, it is clear that for $y \in Y$, $B_Y(y, r) = B_X(x, r) \cap Y$. It is then immediate that any subset S of Y is open in Y (with Y regarded as a metric space in its own right) if and only if $S = Y \cap U$, where U is open in X. Now set $X = \mathbf{K}^n$ and $Y = \pi_J^{-1}(\boldsymbol{z})$ and regard Y as a metric space in its own right with the metric coming from X. The Lemma is essentially the observation we made above, together with the fact that the identification of $\mathbf{K}^r$ with $Y = \pi_J^{-1}(\boldsymbol{z})$ preserves distances (i.e. is an isometry) and hence preserves open sets. Now for the less natural proof.

Proof. This is immediate from (2.2.4.2). In slightly greater detail, let all spaces in consideration be given the Euclidean norm, and for $\varrho > 0$, let $B_{\mathbf{K}^r}(\boldsymbol{x}, \varrho)$ denote the ball of radius ϱ centred at $\boldsymbol{x} \in \mathbf{K}^r$, to distinguish it from $B(\boldsymbol{v}, \varrho)$, the ball of radius ϱ in $\mathbf{K}^n$ centred at $\boldsymbol{v} \in \mathbf{K}^n$. From (2.2.4.2) we see that

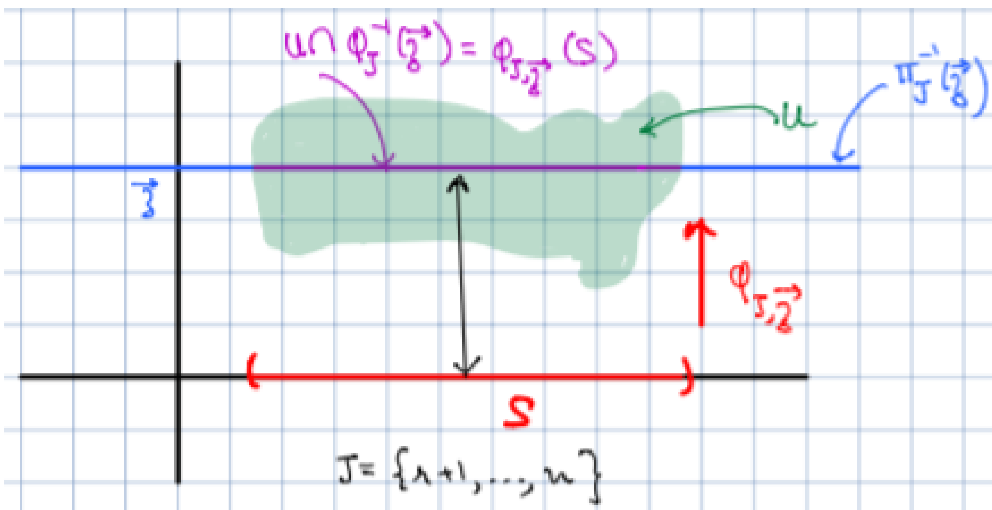
$$(*) \qquad \varphi_{J,\boldsymbol{z}}(B_{\mathbf{K}^r}(\boldsymbol{x}, \varrho) = B(\varphi_{J,\boldsymbol{z}}(\boldsymbol{x}, \boldsymbol{z}), \varrho) \cap \pi_J^{-1}(\boldsymbol{z}).$$

Now suppose S is open in $\mathbf{K}^r$, say $S = \cup_\alpha B_{\mathbf{K}^r}(\boldsymbol{x}_\alpha, r_\alpha)$. Then it is easy to check from $(*)$ that $\varphi_{J,\boldsymbol{z}}(S) = \cup_\alpha B(\varphi_{J,\boldsymbol{z}}(\boldsymbol{x}_{\boldsymbol{\alpha}}, \boldsymbol{z}), r_\alpha) \cap \pi_J^{-1}(\boldsymbol{z})$. It follows that $\varphi_{J,\boldsymbol{z}}(S)$ is open in $\pi_J^{-1}(\boldsymbol{z})$.

Conversely, suppose $\varphi_{J,\boldsymbol{z}}(S)$ is open in $\pi_J^{-1}(\boldsymbol{z})$. Say $\varphi_{J,\boldsymbol{z}}(S) = U \cap \pi_J^{-1}(\boldsymbol{z})$ where U is open in $\mathbf{K}^n$. Then for each point $\boldsymbol{x} \in S$ there exists $r_{\boldsymbol{x}} > 0$ such that

$B(\varphi_{J,\boldsymbol{z}}(\boldsymbol{x},\, \boldsymbol{z}),\, r_{\boldsymbol{x}}) \subset U$. It follows that $B_{\mathbf{K}^r}(\boldsymbol{X},\, r_{\boldsymbol{x}}) \subset S$ and $S = \cup_{\boldsymbol{x}\in S} B_{\mathbf{K}^r}(\boldsymbol{X},\, r_{\boldsymbol{x}})$. This proves that S is open in $\mathbf{K}^r$.

Since $\|\;\|_2 \sim \|\;\|_\infty$, we are done. (See picture below for $J = \{r+1, \dots, n\}$.) $\square$



Definition 2.2.6. Suppose $r + d = n$ where r and d are non-negative integers, and $\mathbf{K}^n = \mathbf{K}^r \times \mathbf{K}^d$ the decomposition in the beginning of Subsection 2.2. A *rectangle* in $\mathbf{K}^n$ with respect to this decomposition is a set of the form $A \times B$, where $A \subset \mathbf{K}^r$ and $B \subset \mathbf{K}^d$.

Theorem 2.2.7. *Let r and d be non-negative integers such that $r + d = n$ and $\mathbf{K}^n = \mathbf{K}^r \times \mathbf{K}^d$ the standard decomposition. A rectangle $A \times B$ with respect to this decomposition is open (respectively closed) in $(\mathbf{K}^n, \|\;\|_2)$ if and only if A and B are open (respectively closed) in $(\mathbf{K}^r, \|\;\|_2)$ and $(\mathbf{K}^d, \|\;\|_2)$ respectively.*

Proof. For the proof we set $I = \{1, \dots, r\}$ and $J = \{r+1, \dots, n\}$. With this J, the map σ in §§2.2.4 is the identity permutation, Φ_σ is the identity map. Moreover, for $\boldsymbol{z} \in \mathbf{K}^d$, $\varphi_{J,\boldsymbol{z}}(\boldsymbol{x}) = (\boldsymbol{x},\, \boldsymbol{z})$, and $\pi_J^{-1}(\boldsymbol{z}) = \mathbf{K}^r \times \{\boldsymbol{z}\}$.

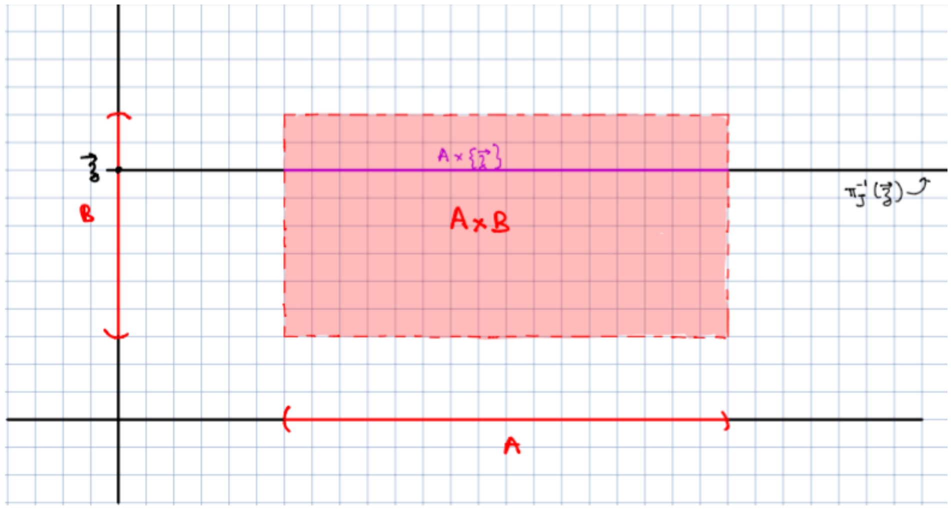
Suppose $U = A \times B$ is open in $(\mathbf{K}^n,\, \|\;\|_2)$. Pick a point $\boldsymbol{z} \in B$. For this $\boldsymbol{z}$, let $\varphi_{J,\boldsymbol{z}} \colon \mathbf{K}^r \to \pi_J^{-1}(\boldsymbol{z})$ be the map in (2.2.4.1). From the description of $\varphi_{J,\boldsymbol{z}}$ and $\pi_J^{-1}(\boldsymbol{z})$ above one sees that

$$\varphi_{J,\boldsymbol{z}}(A) = A \times \{\boldsymbol{z}\} = ((\mathbf{K}^r \times \{\boldsymbol{z}\}) \cap (A \times B) = \mathbf{K}^r \times \{\boldsymbol{z}\}) \cap U = \pi_J^{-1}(\boldsymbol{z}) \cap U.$$

Thus $\varphi_{J,\boldsymbol{z}}(A)$ is open in $\pi_J^{-1}(\boldsymbol{z})$. By Lemma 2.2.5, A is open in $(\mathbf{K}^r, \|\;\|_2)$. By symmetry, B is open in $(\mathbf{K}^d, \|\;\|_2)$. The case where $A \times B$ is closed is dealt with in an identical way.

Conversely, suppose A and B are open. Since π_I and π_J are continuous (see Corollary 2.2.3), therefore by Proposition 2.1.1, $\pi_I^{-1}(A)$ and $\pi_J^{-1}(B)$ are open in $(\mathbf{K}^n,\, \|\;\|_2)$. Now $A \times B = \pi_I^{-1}(A) \cap \pi_J^{-1}(B)$ and hence $A \times B$ is open in $(\mathbf{K}^n,\, \|\;\|_2)$. Using Corollary 2.1.2 and the same argument as we just gave, we see that if A and B are closed then $A \times B$ is closed in $(\mathbf{K}^n,\, \|\;\|_2)$. $\square$

The following picture may be useful:

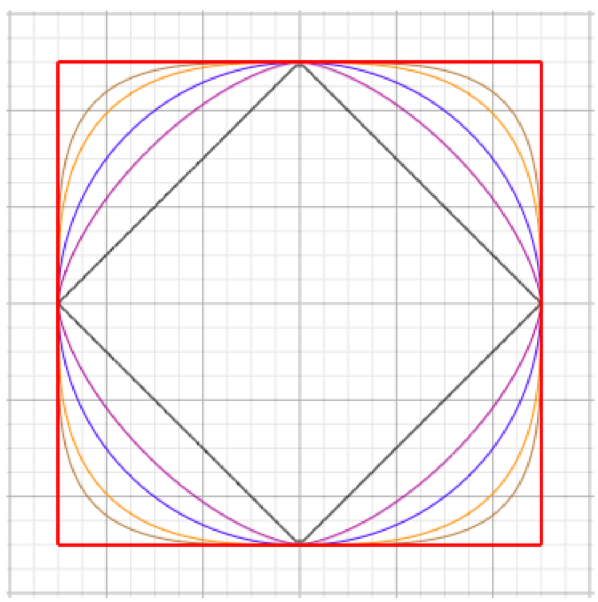


Since $\| \, \|_\infty \sim \| \, \|_2$ we get:

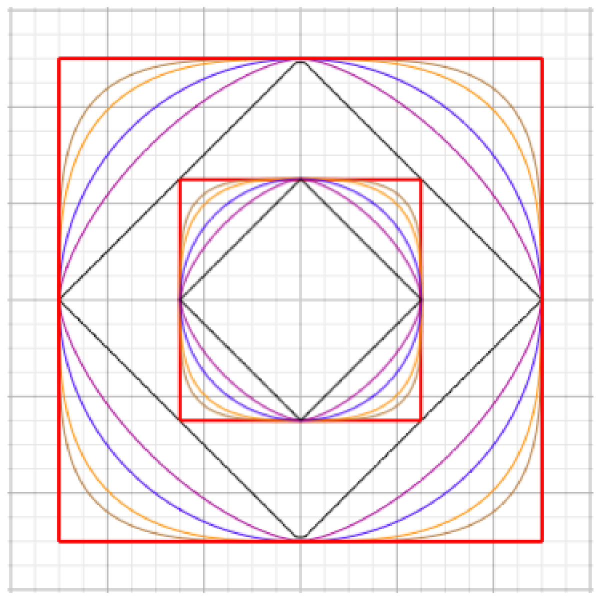
Corollary 2.2.8. *A rectangle $A \times B$ A with respect to the decomposition $\mathbf{K}^n = \mathbf{K}^r \times \mathbf{K}^d$ is open (respectively closed) in $(\mathbf{K}^n, \| \, \|_\infty)$ if and only if A and B are open (respectively closed) in $(\mathbf{K}^r, \| \, \|_\infty)$ and $(\mathbf{K}^d, \| \, \|_\infty)$ respectively.*

3. Some pictures

For a normed space $(X, \| \, \|)$, the unit sphere is defined to be $S(\mathbf{0}, 1) = \{\boldsymbol{x} \in X \mid \|x\| = 1\}$ and $B(\mathbf{0}, 1)$. Here are some unit spheres in $(\mathbf{R}^2, \| \, \|_p)$ for varying p.



The unit balls, are bounded by the unit spheres, and so you get a sense of these too from the picture. The innermost diamond (in black on your browsers and other electronic devices) is for $p = 1$. The outermost square (in red) is for $p = \infty$. The successive spheres, going from the innermost to the outermost are for $p = 1$, $p = 1.5$ (purple), $p = 2$ (blue), $p = 3$ (orange), $p = 4$ (brown), and $p = \infty$. As was pointed out in your tutorial, $\lim_{p\to\infty} \| \|_p = \| \|_\infty$, and the picture gives some evidence for this. Also note that it is easy to fit a $\| \|_\infty$-sphere of radius $1/2$ inside the innermost diamond, and hence all other spheres of radius $1/2$. The picture below shows the various spheres of radius $1/2$ nestling inside $\| \|_1$ unit sphere (and hence all other unit spheres). This is a vivid way of seeing that open sets in each of these norms are the same, since one repeat the process indefinitely inwards and outwards, and see that if $p, q \in [0, \infty)$ then each $B_p(\boldsymbol{x}, r)$ is contained in $B_q(\boldsymbol{x}, s)$ for some s, where B_p is the symbol for balls with respect to $\| \|_p$.



About these notes. These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.