LECTURE 2

Date of Lecture: January 13, 2020

1. **Topology on normed linear spaces**

Throughout $\mathbf{K}$ will denote one of $\mathbf{R}$ or $\mathbf{C}$ and $(X, \|\ \|)$ a normed linear space over $\mathbf{K}$.

For two subsets A, B of a set S, $A \smallsetminus B = \{x \in A \mid x \notin B\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1.1. **Open and closed sets in a normed linear space.** For $\boldsymbol{v} \in X$ and $r > 0$, we define the *open ball* $B(\boldsymbol{v}, r)$ *of radius* r *centred at* $\boldsymbol{v}$ as the set

$$B(\boldsymbol{v}, r) = \{\boldsymbol{x} \in X \mid \|\boldsymbol{x} - \boldsymbol{v}\| < r\}. \tag{1.1.1}$$

Definition 1.1.2. A subset U of X is called *open* (or *open in* X) if it is the union of open balls. A subset S of X is called *closed* (or *closed in* X) if it is the complement of an open set in X, i.e., if $S^{\mathsf{C}} := X \smallsetminus S$ is open in V.

The emptyset $\varnothing$ is considered open (or, if you wish, look at the relation $\varnothing = \cup_{i \in \varnothing} B(\boldsymbol{x}_i, r_i)$). There is also a relative version of open sets given below.

Definition 1.1.3. Let A be a subset of X. A subset U of A is said to be *open in* A if $U = V \cap A$ where V is open in X. A subset C of A is said to be *closed in* A if $A \smallsetminus C$ (i.e. $A \cap C^{\mathsf{C}}$) is open in A. Equivalently, C is closed in A if $C = D \cap A$ where D is closed in X.

Examples 1.1.4. Not all details are worked out in these examples. You may need to work them out—they may be asked in quizzes and test.

(1) X is open for $X = \cup_{r>0} B(\mathbf{0}, r)$.

(2) Let $X = \mathbf{R}$ with the usual absolute value as the norm. Then open subsets of X are the same as what you studied last semester.

(3) The disc in $\mathbf{R}^2$ consisting of points (x, y) such that $x^2 + y^2 < r^2$ is open in $\mathbf{R}^2$ with the usual Euclidean norm $\|\ \|_2$. To see this, note that this set is the same as $B(\mathbf{0}, r)$. On the other hand the set $\overline{B}(\mathbf{0}, r)$ consisting of points (x, y) with $x^2 + y^2 \leq r^2$ is closed in $\mathbf{R}^2$. Why? [Hint: Suppose (u, v) lies outside $\overline{B}(\mathbf{0}, r)$. Then $u^2 + v^2 > r^2$. Let $\varrho = (\sqrt{u^2 + v^2} - r)/2$. Show that $B((u, v), \varrho) \subset V \smallsetminus \overline{B}(\mathbf{0}, r)$. Why does this prove $\overline{B}(\mathbf{0}, r)$ is closed?]

(4) The set $U = \{(x, y) \in \mathbf{R}^2 \mid x > 0\}$ is open in $(\mathbf{R}^2, \|\ \|_2)$. Why? What about $C = \{(x, y) \in \mathbf{R}^2 \mid x \geq 0\}$?

(5) The set $U = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1, \text{ and } x > 0\}$ is open in $\overline{B}(\mathbf{0}, 1)$ where the norm on $\mathbf{R}^2$ is the Euclidean norm.

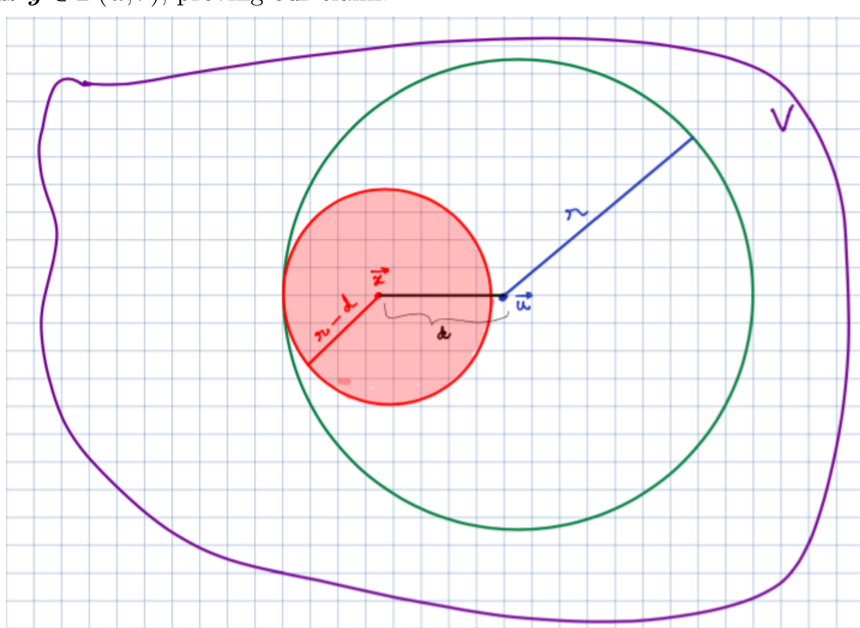
Definition 1.1.5. Let $\boldsymbol{x} \in X$. A *neighbourhood* N of $\boldsymbol{x}$ is a set such that there exists an open set U with the property that $x \in U \subset N$. An *open neighbourhood of* x is a neighbourhood which is open.

1.1.6. One can, as usual, define a relative version of this, namely, for a subset $S \subset X$ and $\boldsymbol{x} \in S$, one can define the notion of a neighbourhood of $\boldsymbol{x}$ in S as well as an the notion of an open neighbourhood of $\boldsymbol{x}$ in S using Definition 1.1.3. I leave the exercise of coming up with the definition(s) to you. And remember you could be asked for these notions in a quiz or a test.

Suppose V is an open set in X and $\boldsymbol{x} \in V$ a point. Then by definition of an open set, there is an open ball $B(\boldsymbol{u}, r) \subset V$ such that $\boldsymbol{x} \in B(\boldsymbol{u}, r)$. Now suppose $\|\boldsymbol{u} - \boldsymbol{x}\| = d$. Then $d < r$. Let $\varrho = r - d$. Then we claim that $B(\boldsymbol{x}, \varrho) \subset B(\boldsymbol{u}, r)$. To see this, suppose $\boldsymbol{y} \in B(\boldsymbol{x}, \varrho)$. By definition, $\|\boldsymbol{y} - \boldsymbol{x}\| < \varrho$. Now

$$\|\boldsymbol{y} - \boldsymbol{u}\| = \|(\boldsymbol{y} - \boldsymbol{x}) - (\boldsymbol{x} - \boldsymbol{u})\| \leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{u}\| < \varrho + d = r.$$

Thus $\boldsymbol{y} \in B(\boldsymbol{u}, r)$, proving our claim.



We have thus shown:

Lemma 1.1.7. *Every point* $\boldsymbol{x}$ *in an open set* V *of* X *is contained in a ball in* V *which has* $\boldsymbol{x}$ *as its centre.*

Theorem 1.1.8. *We have the following:*

(i) *The arbitrary union of open sets is open.*
(ii) *The intersection of two open sets is open*
(iii) *An open subset* W *of an open set* U *is open in* X*.*
(i)′ *The arbitrary intersection of closed sets is closed.*

(ii)′ *The union of two closed sets is closed.*

(iii)′ *A closed subset of a closed set is closed.*

Proof. Let $S = \cup_{\alpha\in A} U_\alpha$ where each U_α is open. For each α we have a collection of open balls $\{B(\boldsymbol{x}_{\alpha,j},\, r_{\alpha,j})\}_{j\in J_\alpha}$ in X such that

$$U_\alpha = \bigcup_{j\in J_\alpha} B(\boldsymbol{x}_{\alpha,j}, r_{\alpha,j}).$$

Then

$$S = \bigcup_{\alpha\in A}\bigcup_{j\in J_\alpha} B(\boldsymbol{x}_{\alpha,j}, r_{\alpha,j}),$$

proving that S is open.

We now prove (ii). Let U and V be open in X and suppose $\boldsymbol{x} \in U\cap V$. By Lemma 1.1.7 we can find open balls $B(\boldsymbol{x}, r_1)$ and $B(\boldsymbol{x}, r_2)$ such that $B(\boldsymbol{x}, r_1) \subset U$ and $B(\boldsymbol{x}, r_2) \subset V$. Let $r_{\boldsymbol{x}} = \min(r_1,\, r_2)$. Then $B(\boldsymbol{x}, r_{\boldsymbol{x}}) \subset B(\boldsymbol{x}, r_1) \cap B(\boldsymbol{x}, r_2) \subset U \cap V$. We therefore have

$$U\cap V = \bigcup_{\boldsymbol{x}\in U\cap V} B(\boldsymbol{x}, r_{\boldsymbol{x}}),$$

whence $U\cap V$ is open.

For (iii) note that since W is open in U, there exists an open set V in X such that $W = U\cap V$. The result follows from (ii).

Part (i)′ is a restatement of (i) by De Morgan's laws of set theory (see https://en.wikipedia.org/wiki/De_Morgan's_laws). For the same reason, (ii)′ is a restatement of (ii). As for (iii)′, let D be a closed subset of the closed set C. Then $D = E\cap C$ where E is closed in X. The result then follows from (i)′. □

1.2. **Limit points.** Let $C \subset X$ and $\boldsymbol{x}\in X$ a limit point of C (see Definition 1.2.2 of Lecture 1). Let U be an open neighbourhood of $\boldsymbol{x}$. We claim that there exists at least one point $\boldsymbol{c}\in C$, with $\boldsymbol{c}\neq \boldsymbol{x}$, such that $\boldsymbol{c}\in U$. In other words, we claim that $U\cap (C \smallsetminus \boldsymbol{x})$ is non-empty. To see this, pick an open ball $B(\boldsymbol{x}, r)$ centred at $\boldsymbol{x}$ such that $B(\boldsymbol{x}, r)\subset U$. According to Lemma 1.1.7 such an open ball always exists. By definition of a limit point of C, there exists a sequence $\{\boldsymbol{c}_n\}$ in C such that $\boldsymbol{c}_n \neq \boldsymbol{x}$ for any $n\in \mathbf{N}$ and such that $\lim_{n\to\infty} \boldsymbol{c}_n = \boldsymbol{x}$. It follows that there exists $N\in\mathbf{N}$ such that $\|\boldsymbol{c}_n - \boldsymbol{x}\| < r$ for $n\geq N$. Let $\boldsymbol{c} = c_N$. Then $\boldsymbol{c}\in C$, $\boldsymbol{c}\neq\boldsymbol{x}$ and $\|\boldsymbol{c}-\boldsymbol{x}\| < r$. This means $\boldsymbol{c}\in B(\boldsymbol{x}, r)\subset U$. Now, $\boldsymbol{c}\neq \boldsymbol{x}$ and $\boldsymbol{c}\in C$ and hence $U\cap C$ contains a point which is not $\boldsymbol{x}$. We have therefore proved our claim.

Conversely, suppose $\boldsymbol{x}\in X$ is such that $U\cap (C\smallsetminus \boldsymbol{x})$ is non-empty for *every open neighbourhood* U *of* $\boldsymbol{x}$. This means $B(\boldsymbol{x}, \frac{1}{n})\cap (C\smallsetminus \boldsymbol{x}) \neq \varnothing$ for every $n\in\mathbf{N}$. For each $n\in\mathbf{N}$ pick $\boldsymbol{c}_n \in B(\boldsymbol{x}, \frac{1}{n})\cap (C\smallsetminus\boldsymbol{x})$. Then $\boldsymbol{c}_n\in C$, $\boldsymbol{c}_n\neq\boldsymbol{x}$ and $\boldsymbol{c}_n\in B(\boldsymbol{x}, \frac{1}{n})$. The last condition means $\|\boldsymbol{c}_n - \boldsymbol{x}\| < \frac{1}{n}$ for every $n\in\mathbf{N}$. It is easy to see that $\boldsymbol{c}_n\to\boldsymbol{x}$ as $n\to\infty$. Indeed, given $\epsilon > 0$, choose $N\in\mathbf{N}$ such that $1/N < \epsilon$. Then $\|\boldsymbol{c}_n - \boldsymbol{x}\| < \frac{1}{n} < \epsilon$ for every $n\geq N$ whence $\lim_{n\to\infty}\boldsymbol{c}_n = \boldsymbol{x}$. Since $\boldsymbol{c}_n\in C\smallsetminus\{\boldsymbol{x}\}$, by definition of a limit point of a set, $\boldsymbol{x}$ is a limit point of C. We thus have part (i) of the following proposition:

Proposition 1.2.1. *Let C be a subset of X.*

(i) *A point $\boldsymbol{x}\in X$ is a limit point of C if and only if every open neighbourhood U of $\boldsymbol{x}$ contains a point of C which is not equal to $\boldsymbol{x}$.*

(ii) *Let* $\overline{C}$ *be the closure of* C *as defined in 1.2.2 of Lecture 1. Then*

$$\overline{C} = \{\boldsymbol{x} \in X \mid \boldsymbol{x} = \lim_{n\to\infty} \boldsymbol{c}_n \text{ for some sequence } \{\boldsymbol{c}_n\} \text{ in } C\}.$$

(iii) *We also have*

$$\overline{C} = \{\boldsymbol{x} \in X \mid U \cap C \neq \varnothing \text{ for every open neighbourhood } U \text{ of } \boldsymbol{x}\}.$$

Proof. Part (i) was proved in the discussion above the statement of the Proposition.

For (ii) let $S = \{\boldsymbol{x} \in X \mid \boldsymbol{x} = \lim_{n\to\infty} \boldsymbol{c}_n$ for some sequence $\{\boldsymbol{c}_n\}$ in $C\}$. If $\boldsymbol{c} \in C$ consider the constant sequence $\{\boldsymbol{c}_n\}$ with each $\boldsymbol{c}_n = c$. It follows that $\boldsymbol{c} \in S$, for $\boldsymbol{c} = \lim_{n\to\infty} \boldsymbol{c}_n$ Thus $C \subset S$. As in [Lecture 1, Definition 1.2.2] , let C' consist of limit points of C. Clearly $C' \subset S$. We claim $S = C \cup C'$. By definition of C', $S \setminus C = C' \setminus C$. It follows that $S = C \cup C'$ and the right side is by definition $\overline{C}$.

Part (iii) is proved in a very similar way. Let

$$T = \{\boldsymbol{x} \in X \mid U \cap C \neq \varnothing \text{ for every open neighbourhood } U \text{ of } \boldsymbol{x}\}.$$

Clearly $C \subset T$. By part (i), $C' \subset T$, and $T \setminus C = C' \setminus C$. The rest is exactly the same as the proof of (ii), and one sees that $T = C \cup C' = \overline{C}$. □

1.3. **The closure of a set.** Let $S \subset X$. We claim that $\overline{S}$ is closed. Indeed, if $\boldsymbol{y} \notin \overline{S}$, then according to Proposition 1.2.1 (iii), there is an open neighbourhood $V_{\boldsymbol{y}}$ of $\boldsymbol{y}$ such that $S \cap V_{\boldsymbol{y}} = \varnothing$, i.e. $V_{\boldsymbol{y}} \subset S^{\complement}$. In fact for any $\boldsymbol{z} \in V_{\boldsymbol{y}}$, we have $V_{\boldsymbol{y}}$ is an open neighbourhood of $\boldsymbol{z}$ which does not intersect S. Hence $\boldsymbol{z} \notin \overline{S}$ for any $\boldsymbol{z} \in V_{\boldsymbol{y}}$, and this in turn means that $V_{\boldsymbol{y}} \subset \overline{S}^{\complement}$. It follows that

$$\overline{S}^{\complement} = \bigcup_{\boldsymbol{y} \in \overline{S}^{\complement}} V_{\boldsymbol{y}},$$

whence, by definition of a closed set, $\overline{S}$ is closed.

Next, suppose F is closed and $F \supset S$. Let $V = F^{\complement}$. Since $V \cap S = \varnothing$, by Proposition 1.2.1 (iii), no point of V lies in $\overline{S}$, i.e. $V \subset \overline{S}^{\complement}$. Thus $\overline{S} \subset F$. Thus $\overline{S}$ is the *smallest closed set containing* S, for we have shown that if F is closed and contains S then it contains $\overline{S}$. If $\mathscr{F}$ is the collection of closed sets F such that $F \supset S$, then from what we have shown, we have $\overline{S} = \cap_{F\in\mathscr{F}} F$.

We point out that if S is closed then $S = \overline{S}$. Indeed if S is closed, then $S \in \mathscr{F}$, and hence $\overline{S} \subset S$, giving $S = \overline{S}$. Conversely, if $S = \overline{S}$, then clearly S is closed, since $\overline{S}$ is closed. Finally, since $\overline{S} = S \cup S'$, where S' is the set of limit points of S, the relation $S = \overline{S}$ holds if and only if $S' \subset S$. In other words S is closed if and only if it contains all its limit points.

We have proven the following:

Theorem 1.3.1. *Let* $S \subset X$. *Then*

(i) $\overline{S}$ *is closed.*
(ii) S *is closed if and only if* $S = \overline{S}$.
(iii) S *is closed if and only if it contains all its limit points.*
(iv) $\overline{S}$ *is the smallest closed set containing* S, *i.e. if* F *is a closed set containing* S, *then* $\overline{S} \subset F$. *In particular if* $\mathscr{F} = \{F \mid F \text{ is a closed set and } S \subset F\}$, *then*

$$\overline{S} = \bigcap_{F\in\mathscr{F}} F.$$

2. Norms on $\mathbf{K}^n$

We will see later in the course that on a finite dimensional space X over $\mathbf{K}$, all norms give the same collection of open sets and the same collection of closed sets and convergence of a sequence in one norm implies its convergence in all other norms on X. The proof of that will have to wait until we do the Heine-Borel theorem on $\mathbf{K}^n$. However, we can do a special case right now.

2.1. **The spaces $(\mathbf{K}^n, \|\ \|_\infty)$ and $(\mathbf{K}^n, \|\ \|_2)$.** Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{K}^n$. Then

$$|x_i| \le (|x_1|^2 + \cdots + |x_n|)^{\frac{1}{2}} = \|\boldsymbol{x}\|_2$$

for $i = 1, \dots, n$. It follows that $\max_{i=1,\dots,n} |x_i| \le \|\boldsymbol{x}\|_2$. In other words

$$\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_2. \tag{2.1.1}$$

One can show easily that this proves that if $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_2)$ then $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_\infty)$. The proof is given in a more general situation in the proof Lemma 2.1.5 below.

On the other hand, $|x_i|^2 \le \|\boldsymbol{x}\|_\infty^2$ for every $i = 1, \dots, n$, whence on summing over i we get $\sum_{i=1}^n |x_i|^2 \le n\|\boldsymbol{x}\|_\infty^2$. In other words

$$\|\boldsymbol{x}\|_2 \le \sqrt{n}\|\boldsymbol{x}\|_\infty. \tag{2.1.2}$$

This inequality shows that if $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_\infty)$, then $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{x}$ in $(\mathbf{K}^n, \|\ \|_2)$. (See proof of Lemma 2.1.5 below.)

Definition 2.1.3. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a $\mathbf{K}$-vector space X are said to be *equivalent* if there exist constants $c, C > 0$ such that

$$c\|\boldsymbol{x}\|' \le \|\boldsymbol{x}\| \le C\|\boldsymbol{x}\|' \qquad (\boldsymbol{x} \in X).$$

If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms, we write $\|\cdot\| \sim \|\cdot\|'$.

It is clear that $\sim$ is an equivalence relation on the set of norms on X.

Example 2.1.4. From (2.1.1) and (2.1.2) we see that on $\mathbf{K}^n$, $\|\ \|_\infty \sim \|\ \|_2$. More generally, we will show later in this course that if $\|\ \|$ and $\|\ \|'$ are two norms on $\mathbf{K}^n$ then they are equivalent.

Lemma 2.1.5. *Let $\|\ \|$ and $\|\ \|'$ be equivalent norms on the $\mathbf{K}$-vector space X, $\{\boldsymbol{x}_m\}$ a sequence in X, and S a subset of X. Then*

(i) *The sequence $\{\boldsymbol{x}_n\}$ converges to $\boldsymbol{x} \in X$ with respect to $\|\ \|$ if and only if it does so with respect to $\|\ \|'$.*

(ii) *S is open (respectively closed) with respect to $\|\ \|$ if and only if it is open (respectively closed) with respect to $\|\ \|'$.*

Proof. Let $c, C > 0$ be such that

$$c\|\ \|' \le \|\ \| \le C\|\ \|'. \tag{$*$}$$

Suppose $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ in $(X, \|\ \|)$. Given $\epsilon > 0$ we have $N \in \mathbf{N}$ such that $\|\boldsymbol{x}_n - \boldsymbol{x}\| < c\epsilon$ for $n \ge N$. This means $c\|\boldsymbol{x}_n - \boldsymbol{x}\|' \le \|\boldsymbol{x}_n - \boldsymbol{x}\| < c\epsilon$ for $n \ge N$, i.e. $\|\boldsymbol{x}_n - \boldsymbol{x}\|' < \epsilon$ for $n \ge N$, showing that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ in $(X, \|\ \|')$. The converse is follows because $\sim$ is an equivalence relation. This proves (i).

To prove part (ii), it is enough to prove that S is open in with respect to $\|\ \|$ if and only if it is so with respect to $\|\ \|'$, for the case where S is closed then follows by taking complements. To prove this, we need notations. Let $B(\boldsymbol{x}, r)$

and $B'(\boldsymbol{x}, r)$ denote the open balls of radius r centred at $\boldsymbol{x}$ with respect to $\| \|$ and $\| \|'$ respectively. In other words $B(\boldsymbol{x}, r) = \{\boldsymbol{z} \in X \mid \|\boldsymbol{z} - \boldsymbol{x}\| < r\}$ and $B'(\boldsymbol{x}, r) = \{\boldsymbol{z} \in X \mid \|\boldsymbol{z} - \boldsymbol{x}\|' < r\}$. From $(*)$ we get

$$(\dagger) \qquad B'(\boldsymbol{x}, r) \subset B(\boldsymbol{x}, Cr), \quad \text{and} \quad B(\boldsymbol{x}, r) \subset B'(\boldsymbol{x}, r/c).$$

Let S be open with respect to $\| \|$. According to Lemma 1.1.7, for each $\boldsymbol{x} \in S$ there is an open ball $B(\boldsymbol{x}, r_{\boldsymbol{x}}) \subset S$. By $(\dagger)$ we have $B'(\boldsymbol{x}, r_{\boldsymbol{x}}/C) \subset S$. It follows that $S = \cup_{\boldsymbol{x} \in S} B'(\boldsymbol{x}, r_{\boldsymbol{x}}/C)$ and hence S is open with respect to $\| \|'$. By symmetry, if S is open with respect to $\| \|'$ it is so with respect to $\| \|$. This proves (ii). □

Remark 2.1.6. From (2.1.1) and (2.1.2) we see that that the open (respectively closed) sets of $(\mathbf{K}^n, \| \|_\infty)$ coincide with those of $(\mathbf{K}^n, \| \|_2)$. We also see that a sequence $\{\boldsymbol{x}_m\}$ converges to $\boldsymbol{x}$ in $(\mathbf{K}^n, \| \|_\infty)$ if and only if it converges to $\boldsymbol{x}$ in $(\mathbf{K}^n, \| \|_2)$.

Corollary 2.1.7. *Let $\{\boldsymbol{x}_m\}$ be a sequence in $\mathbf{K}^n$, say $\boldsymbol{x}_m = (x_{1m}, \dots, x_{nm})$. Then*

$$\lim_{m \to \infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$$

with respect to $\| \|_2$ if and only if

$$\lim_{m \to \infty} x_{im} = x_i$$

for $i = 1, \dots, n$.

Proof. By Remark 2.1.6 we have, $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect $\| \|_2$ if and only if $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect to $\| \|_\infty$. Now $\lim_{m\to\infty} \boldsymbol{x}_m = (x_1, \dots, x_n)$ with respect to $\| \|_\infty$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$\max_{i=1,\dots,n} |x_{im} - x_i| < \epsilon \qquad (m \geq N).$$

The above it true if and only if

$$|x_{im} - x_i| < \epsilon \qquad (m \geq N; i = 1, \dots, n).$$

i.e. if and only if $\lim_{m\to\infty} x_{im} = x_i$ for every $i \in \{1, \dots, n\}$. □

2.1.8. Ending Remark. If $\boldsymbol{x} \in X$, then $\{\boldsymbol{x}\}$ is closed in X. Indeed, if $\boldsymbol{y} \in X \smallsetminus \{\boldsymbol{x}\}$, then $\|\boldsymbol{y} - \boldsymbol{x}\| > 0$, and hence a ball of radius $\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{x}\|$ centred at $\boldsymbol{y}$ will not intersect $\{\boldsymbol{x}\}$.

About these notes. These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.