LECTURE 18

Date of Lecture: March 30, 2020

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \|_2$ and we will simply denote it as $\| \|$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Lagrange's Multipliers

1.1. One constraint. Recall the following from Homework 6. Let U be an open subset of $\mathbf{R}^n$ and $h\colon U \to \mathbf{R}$, $f\colon U \to \mathbf{R}$ two $\mathscr{C}^1$ functions on U. Let $c \in h(U)$ and let M be the level set $M = \{\boldsymbol{x} \in U \mid h(\boldsymbol{x}) = c\}$. We say $f|_M$ has a *local maximum* at $\boldsymbol{v} \in M$ if there exists an open neighbourhood W of $\boldsymbol{v}$ in U such that $f(\boldsymbol{x}) \leq f(\boldsymbol{v})$, for all $\boldsymbol{x} \in W \cap M$. It is said to have a *local minimum* at $\boldsymbol{v}$ if there exists an open neighbourhood W of $\boldsymbol{v}$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{v})$ for all $\boldsymbol{x} \in W \cap M$. Finally, $f|_M$ is said to have a *local extremum* at $\boldsymbol{v}$ if it has either a local maximum or a local minimum at $\boldsymbol{v}$.

We assume for the rest of this subsection that h' does not vanish at any point of M.

According to Problems 6 and 7of Homework 6, under the above assumption, the following is true:

(a) M has a tangent space at each of its points.
(b) If $f|_M$ has a local extremum at $\boldsymbol{v} \in M$ then $\boldsymbol{\nabla} f(\boldsymbol{v})$ is orthogonal to the tangent space of M at $\boldsymbol{v}$.
(c) Suppose h' does not vanish at any point of M and suppose $f|_M$ has a local extremum at $\boldsymbol{v} \in M$. Show that there is a unique scalar $\lambda \in \mathbf{R}$ such that $\boldsymbol{\nabla} f(\boldsymbol{v}) = \lambda \boldsymbol{\nabla} h(\boldsymbol{v})$.

The extreme points and extreme values of $f|_M$ can be regarded as the *maxima and minima of f subject to the constraint $h = c$*. The condition $h = c$ is called the *constraint* and the problem of maximising or minimising the function f subject to this constraint is clearly the same as maximising or minimising $f|_M$.

Suppose $\boldsymbol{v}$ is a local extremum of $f|_M$. We know (from (c) above) that there is a unique scalar $\lambda \in \mathbf{R}$ such that $\boldsymbol{\nabla} f(\boldsymbol{v}) = \lambda \boldsymbol{\nabla} h(\boldsymbol{v})$. The scalar λ is called the *Lagrange multiplier* at $\boldsymbol{v}$ for f subject to the constraint $h = c$.

Examples 1.1.1.

1. Let E be the ellipse in the xy-plane given by $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and let $f(x, y) = xy$. Let us use the method of Lagrange's multipliers to find the points on E at which $f|_E$ has a local extremum.

Let $h(x, y) = \frac{x^2}{9} + \frac{y^2}{4}$. Then the Jacobian of h is the matrix $Jh(x, y) = [\frac{2}{9}x \quad \frac{1}{2}y]$. This is zero if and only if $(x, y) = \mathbf{0}$. It follows that Jh is never zero on E (for $\mathbf{0} \notin E$), i.e. h' does not vanish on E. Our constraint is $h = 1$.

Suppose $(x, y) \in E$ is a point of local extremum for $f|_E$. Since $\boldsymbol{\nabla} f(x, y) = (y, x)$ and $\boldsymbol{\nabla} h(x, y) = (\frac{2}{9}x, \frac{1}{2}y)$, we have

$$(y, x) = \lambda(\tfrac{2}{9}x, \tfrac{1}{2}y).$$

where λ is a Lagrange multiplier. Thus

$$y = \lambda(\tfrac{2}{9}x) \quad \text{and} \quad x = \lambda(\tfrac{1}{2}y).$$

It follows that $x = 0$ if and only if $y = 0$, and since $(x, y) \in E$, this is not possible. In particular, it is safe to divide by x or y. We therefore have

$$\frac{9y}{2x} = \frac{2x}{y} = \lambda.$$

Thus

$$(*) \qquad 4x^2 - 9y^2 = 0.$$

On the other hand the equation of E can be re-written as

$$(**) \qquad 4x^2 + 9y^2 = 36.$$

Thus $y^2 = 2$ and $x^2 = \frac{9}{2}$ and the solutions of the system of equations given by $(*)$ and $(**)$ are $(3/\sqrt{2}, \sqrt{2})$, $(3/\sqrt{2}, -\sqrt{2})$, $(-3/\sqrt{2}, -\sqrt{2})$, and $(3/\sqrt{2}, -\sqrt{2})$. The data can be arranged as follows.

$(3/\sqrt{2}, \sqrt{2})$	$f(\sqrt{2}, 3/\sqrt{2}) = 3$	$\lambda = 3$
$(3/\sqrt{2}, -\sqrt{2})$	$f(-\sqrt{2}, 3/\sqrt{2}) = -3$	$\lambda = -3$
$(-3/\sqrt{2}, -\sqrt{2})$	$f(-\sqrt{2}, -3/\sqrt{2}) = 3$	$\lambda = 3$
$(3/\sqrt{2}, \sqrt{2})$	$f(\sqrt{2}, -3/\sqrt{2}) = -3$	$\lambda = -3$

Since E is compact, f attains a maximum and a minimum on E, and the points where these are attained must be amongst the above four points. Inspecting the table above we see that $(3/\sqrt{2}, \sqrt{2})$ and $(-3/\sqrt{2}, -\sqrt{2})$ are points of maxima for f and the 3 is the maximum value of $f|_E$. Again, inspecting the table, we see that $(-3/\sqrt{2}, \sqrt{2})$ and $(3/\sqrt{2}, -\sqrt{2})$ are points of minima and the minimum value of f on E is -3.

In doing the above problem using Lagrange's multipliers, we eliminated λ and solved for x and y. But that is not the only way to do this. We could have instead tried to solve for the Lagrange multiplier λ and this is possible (details are left to you). If we had followed that strategy, we would have found $\lambda = \pm 3$. Using this one could then solve for x and y.

The problem can also be done by parameterising E. This is not always possible, but in this example we can. Indeed $x = 3\cos\theta$, $y = 2\sin\theta$, $\theta \in [0, 2\pi)$ is a parameterisation of E. Now

$$f(3\cos\theta, 2\sin\theta) = 6\cos\theta\sin\theta = 3\sin 2\theta.$$

The maximum value of $3\sin 2\theta$ is 3, and this is achieved when $2\theta = \pi/2, 5\pi/2$, i.e. when $\theta = \pi/4$ and $5\pi/4$. Now $3\cos\pi/4 = 3/\sqrt{2}$, $2\sin\pi/4 = \sqrt{2}$, $3\cos 5\pi/4 = -3/\sqrt{2}$, and $2\sin 5\pi/4 = -\sqrt{2}$, giving us the same two points of maxima that we had earlier. Similarly the minimum value of $3\sin 2\theta$ is -3 and this is achieved when $\theta = 3\pi/4$ and $9\pi/4$, giving us the points of minima that we had earlier.

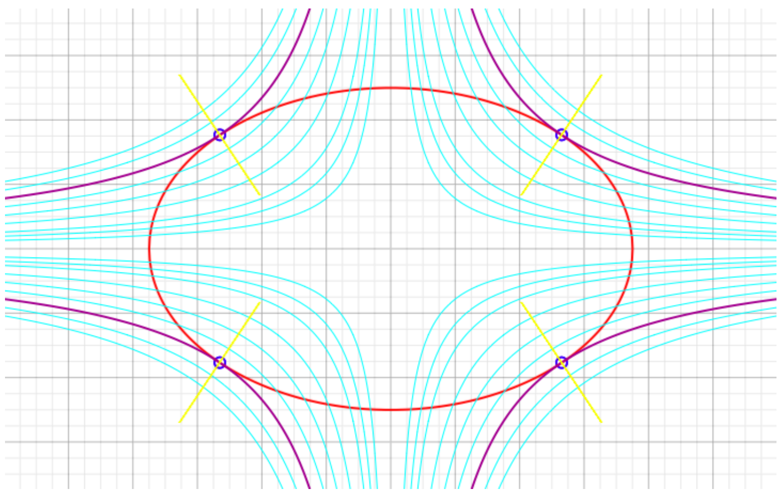

Figure 1. The hyperbolas (in blue and purple) are the level sets of $f(x, y) = xy$ and the (red) ellipse is E. The four points extreme points are marked, one in each quadrant, as well as the normal directions to E at the points (in yellow). The purple level sets are ones which are tangential to E at some point, and these points are where the local extrema of $f|_E$ occur. Note the manner in which the normal direction, and hence the tangent direction, is shared by E and the level curves of f at the constrained local extrema of f.

2. Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be the function $f(x,y) = x^2 - y^2$. Let us maximise and minimise on the unit circle C centred at $\mathbf{0}$, i.e. on the curve given by $x^2 + y^2 = 1$. A look at the picture below tells us that the extrema occur at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. We will do the computations necessary below the figure.

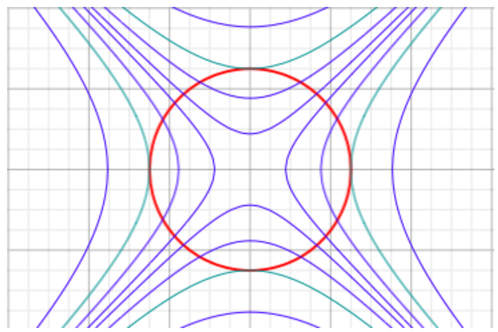

FIGURE 2. Level curves of $f(x,y) = x^2 - y^2$ are in blue, except the ones tangential to the constraint locus $x^2+y^2 = 1$. Those level curves are in green.

The equation involving Lagrange's multipliers is $(2x, -2y) = \lambda(2x, 2y)$. Thus any local extremum of $f|_C$ must be of the form (α, β) where

$$\alpha = \lambda\alpha \quad \text{and} \quad \beta = -\lambda\beta$$

for some Lagrange multiplier λ. Since $\mathbf{0} \notin C$ and $(\alpha, \beta) \in C$, at least one of α or β is non zero. If $\alpha \neq 0$, then the above equations show that $\lambda = 1$ whence $\beta = -\beta$, i.e. $\beta = 0$. Since $\alpha^2 + \beta^2 = 1$, this gives $\alpha = \pm 1$. Thus $\boldsymbol{p}_1 = (1,0)$ and $\boldsymbol{p}_2 = (-1,0)$ are possible points of local extrema of $f|_C$. Similarly, if $\beta \neq 0$, then $\lambda = -1$ and $\alpha = 0$, giving $\beta = \pm 1$. Thus $\boldsymbol{q}_1 = (0,1)$ and $\boldsymbol{q}_2 = (0,-1)$ are the other possible points of local extrema. Computing f at these four points, and using the fact that C is compact (whence $f|_C$ has a maximum or a minimum), we see that $f|_C$ attains its maximum of 1 at $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, and its minimum value of -1 at $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

3. Let us find the points on the sphere S given by $x^2 + y^2 + z^2 = 9$ which are closest and farthest away to the point $\boldsymbol{u} = (3, -1, 2)$. The distance of a point $(x,y,z) \in S$ from $\boldsymbol{u}$ is $\Delta(x,y,z) = \sqrt{(x-3)^2 + (y+1)^2 + (z-2)^2}$. We have to maximise and minimise Δ subject to the constraint $x^2 + y^2 + z^2 = 9$. It is easier to maximise and minimise $\Delta^2 = (x-3)^2 + (y+1)^2 + (z-2)^2$ subject to the same constraint and that is what we will do. So let $f = \Delta^2$ (on $\mathbf{R}^3$) and $h(x,y,z) = x^2 + y^2 + z^2$. The constraint is $h = 9$. The Lagrange's multiplier condition is $\boldsymbol{\nabla} f(x,y,z) = \lambda \boldsymbol{\nabla} h(x,y,z)$, with $h = 9$. This gives the following four equations

$$\begin{cases} x - 3 = \lambda x \\ y + 1 = \lambda y \\ z - 2 = \lambda z \\ x^2 + y^2 + z^2 = 9 \end{cases} \tag{1}$$

From the first of these equations we have $(1-\lambda)x = 3$. We would like to divide by $1-\lambda$, but we have to argue that $\lambda \neq 1$. From the second equation it is clear that $\lambda \neq 1$, and also from the third. Thus

$$x = \frac{3}{1-\lambda}, \quad y = \frac{-1}{1-\lambda}, \quad \text{and} \quad z = \frac{2}{1-\lambda}.$$

From the fourth equation, namely $x^2+y^2+z^2 = 9$, we get $\frac{14}{(1-\lambda)^2} = 9$, i.e.

$$1-\lambda = \pm\sqrt{\tfrac{14}{9}}.$$

Thus the solutions to the system of equation (1) are

$$\boldsymbol{v} = \left(3\sqrt{\tfrac{9}{14}}, -\sqrt{\tfrac{9}{14}}, 2\sqrt{\tfrac{9}{14}}\right) \quad \text{and} \quad \boldsymbol{w} = \left(-3\sqrt{\tfrac{9}{14}}, \sqrt{\tfrac{9}{14}}, -2\sqrt{\tfrac{9}{14}}\right).$$

It is worth noting that $\boldsymbol{v} = \sqrt{\frac{9}{14}}\boldsymbol{u}$ and $\boldsymbol{w} = -\sqrt{\frac{9}{14}}\boldsymbol{u}$. Thus

$$\Delta(\boldsymbol{v}) = \|\boldsymbol{v}-\boldsymbol{u}\| = \left(1-\sqrt{\tfrac{9}{14}}\right)\|\boldsymbol{u}\| = \sqrt{14}-3$$

and

$$\Delta(\boldsymbol{w}) = \|\boldsymbol{w}-\boldsymbol{u}\| = \left(1+\sqrt{\tfrac{9}{14}}\right)\|\boldsymbol{u}\| = \sqrt{14}+3.$$

It follows that $\boldsymbol{w}$ is the point on S furthest away from $\boldsymbol{u}$, and is at a distance $\sqrt{14}+3$ from $\boldsymbol{u}$ and $\boldsymbol{v}$ is the point on S closest to $\boldsymbol{u}$, and is at a distance $\sqrt{14}-3$ from $\boldsymbol{u}$.

About these notes. This lecture was supposed to be given on March 30, 2020. Classes got suspended on March 17 because of the coronavirus COVID 19 pandemic, and all teaching moved online. These course notes are a reasonably faithful record of the lectures given (before the shutdown) at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.

References

[MT] J.E. Marsden and A.J. Tromba, *Vector Calculus*, Fifth edition, W.H. Freeman and Company, New York, 2003.

[St] J. Stewart, *Calculus*, Sixth edition, Thomson Brooks/Cole, 2008.