LECTURE 17

Date of Lecture: March 25, 2020

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \|_2$ and we will simply denote it as $\| \|$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Local Maxima and Minima

1.1. **Critical points.** Let $f \colon U \to \mathbf{R}$ be a $\mathscr{C}^1$ map, where U is an open subset of $\mathbf{R}^n$. A point $\boldsymbol{a}$ in U is called a *critical point* of f if $f'(\boldsymbol{a}) = 0$. There are slightly more general definitions of critical points, where we relax the $\mathscr{C}^1$ condition and require that either $f'(\boldsymbol{a}) = 0$ or $f'(\boldsymbol{a})$ does not exist. However, in this course, we will not need the more general definition (as far as I can see, at the moment).

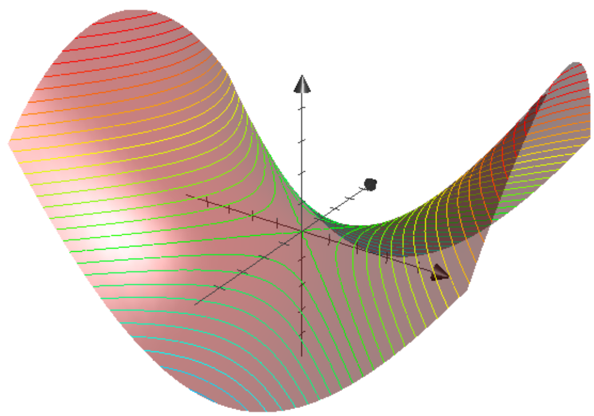
If $\boldsymbol{a} = (a_1, \dots, a_n)$ is a point of local extremum for f, and ℓ is a line passing through $\boldsymbol{a}$, then $\boldsymbol{a}$ is a point of local extremum for $f|_{\ell \cap U}$. This means, from one variable calculus, $D_{\boldsymbol{u}} f(\boldsymbol{a}) = 0$ for every unit vector $\boldsymbol{u}$. In particular $\frac{\partial f(\boldsymbol{x})}{\partial x_i}|_{\boldsymbol{x}=\boldsymbol{a}} = 0$ for $i = 1, \dots, n$, i.e. $f'(\boldsymbol{a}) = 0$. Thus points of local extrema for f are critical points of f. The converse need not be true as we know from one variable Calculus (the standard example is the critical point $x = 0$ for $f(x) = x^3$).

Definition 1.1.1. Let the function f above be $\mathscr{C}^2$. A point $\boldsymbol{a} \in U$ is said to be a *nondegenerate critical point* of f if it is a critical point of f and the Hessian matrix at $\boldsymbol{a}$, $H(f)(\boldsymbol{a})$, is non-singular. The *index* $\sigma(\boldsymbol{a})$ of a nondegenerate critical point $\boldsymbol{a}$ of f is the number of negative eigenvalues of $H(f)(\boldsymbol{a})$.

Assume f is $\mathscr{C}^2$ then $H(f)$ is a symmetric matrix of functions, and its value at any point on U is a symmetric matrix. This follows from the equality of mixed partials “under permutations” of the order of differentiation that was proved earlier in this course. If $\boldsymbol{a}$ is a nondegenerate critical point of f, then, clearly, $H(f)(\boldsymbol{a})$ is positive definite if and only if its index is zero, and it is negative definite if and only if its index is n.

Definition 1.1.2. A nondegenerate critical point $\boldsymbol{a}$ of f is said to be a *saddle point* if $0 < \sigma(\boldsymbol{a}) < n$, i.e. if $H(f)(\boldsymbol{a})$ is indefinite.

Remark 1.1.3. If $n = 2$, the the graph of f near a saddle point looks like this:



In the above, the origin is a saddle point. Note that the contour lines (the level curves) which have been drawn on the surface seem to cross each other at the saddle point. That is often the way to detect saddle points from contours.

Theorem 1.1.4. *Let f be a real valued $\mathscr{C}^3$ function on an open subset U of $\mathbf{R}^n$ and $\boldsymbol{a}$ a nondegenerate critical point of f. Let H be the Hessian of f at $\boldsymbol{a}$. Then*

(a) *f has a local minimum at $\boldsymbol{a}$ if and only if H is positive definite; and*
(b) *f has a local maximum at $\boldsymbol{a}$ if and only if H is negative definite.*

Proof. Since $f'(\boldsymbol{a}) = 0$, for $\boldsymbol{x}$ sufficiently close to $\mathbf{0}$, Taylor's expansion gives us

$$(*) \qquad f(\boldsymbol{a}+\boldsymbol{x}) - f(\boldsymbol{a}) = \tfrac{1}{2}\boldsymbol{x}^t H\boldsymbol{x} + r(\boldsymbol{x})$$

where $r(\boldsymbol{x})/\|\boldsymbol{x}\|^2 \to 0$ as $\boldsymbol{x} \to \mathbf{0}$ (see Theorem 1.4.2 of Lecture16).

Since the quadratic form $Q\colon \mathbf{R}^n \to \mathbf{R}$ given by $\boldsymbol{x} \mapsto \boldsymbol{x}^t H\boldsymbol{x}$ is continuous (in fact $\mathscr{C}^\infty$), it attains a maximum and a minimum on the unit sphere $S = S^{n-1}$ of $\mathbf{R}^n$. Let

$$\mu = \min_{\boldsymbol{x}\in S} Q(\boldsymbol{x}).$$

Suppose H is positive definite. Then $Q(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in S$ and hence $\mu > 0$. Moreover, since for non-zero $\boldsymbol{x}$, $\boldsymbol{x}/\|\boldsymbol{x}\| \in S$, we have

$$\frac{Q(\boldsymbol{x})}{\|\boldsymbol{x}\|^2} \geq \mu \qquad (\mathbf{0} \neq \boldsymbol{x} \in \mathbf{R}^n).$$

Since $r(\boldsymbol{x})/\|\boldsymbol{x}\|^2 \to 0$ as $\boldsymbol{x} \to \mathbf{0}$, there exists $\delta > 0$ such that $|r(\boldsymbol{x})|/\|\boldsymbol{x}\|^2 < \mu$ for $0 < \|\boldsymbol{x}\| < \delta$. It follows that

$$\frac{Q(\boldsymbol{x}) + r(\boldsymbol{x})}{\|\boldsymbol{x}\|^2} > 0 \qquad (0 < \|\boldsymbol{x}\| < \delta).$$

By $(*)$ this means that

$$f(\boldsymbol{a} + \boldsymbol{x}) - f(\boldsymbol{a}) > 0 \qquad (0 < \|\boldsymbol{x}\| < \delta).$$

Thus f has a local minimum at $\boldsymbol{a}$ whenever H is positive definite.

Conversely, suppose f has a local minimum at $\boldsymbol{a}$. Let $\boldsymbol{u}$ be an eigenvector of H of unit length with eigenvalue λ. Let ℓ be the line of support of $\boldsymbol{u}$, i.e. let $\ell = \{t\boldsymbol{u} \mid t \in \mathbf{R}\}$. Then $f|_{(\ell+\boldsymbol{a})\cap U}$ has a local minimum at $\boldsymbol{a}$. Using one variable calculus,[1] this means that

$$(\dagger) \qquad D^2_{\boldsymbol{u}} f(\boldsymbol{a}) \geq 0.$$

Since $D_{\boldsymbol{u}} g(\boldsymbol{x}) = \sum_{i=1}^n u_i (D_i g)(\boldsymbol{x})$ for all $\mathscr{C}^1$ functions g on U and all $\boldsymbol{x} \in U$, it is clear that

$$D^2_{\boldsymbol{u}} f(\boldsymbol{a}) = \boldsymbol{u}^t H \boldsymbol{u}.$$

Now

$$\boldsymbol{u}^t H \boldsymbol{u} = \boldsymbol{u}^t(\lambda \boldsymbol{u}) = \lambda\langle \boldsymbol{u}, \boldsymbol{u}\rangle = \lambda,$$

i.e. $D^2_{\boldsymbol{u}} f(\boldsymbol{a}) = \lambda$, whence $\lambda \geq 0$ by $(\dagger)$. Now, H is non-degenerate, and so λ cannot be zero (if a square matrix has 0 as an eigenvalue, then it has a non-trivial null space, and hence is not invertible). Thus $\lambda > 0$. Since all the eigenvalues of H are positive, we are done. This proves (a).

Part (b) follows since H is negative definite if and only if $-H$ is positive definite, and f has a local maximum at $\boldsymbol{a}$ if and only if $-f$ has a local minimum at $\boldsymbol{a}$. □

For $n = 2$ we have the following easy test. We state it in the form that is found in most of the thick 300 to 1000 page Calculus books (which books should not be disparaged — they have their plus points — and if you have one, stick to it like glue).

Corollary 1.1.5. *Let U be an open subset of $\mathbf{R}^2$, f a $\mathscr{C}^3$ real valued function on U, and $\boldsymbol{a}$ a nondegenerate critical point. Let D be the determinant of the Hessian of f at $\boldsymbol{a}$, i.e.*

$$D = \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a})\frac{\partial^2 f}{\partial y^2}(\boldsymbol{a}) - \left\{\frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{a})\right\}^2.$$

(a) *f has a local extremum at $\boldsymbol{a}$ if and only if $D > 0$. In this case:*
  (i) *f has a local minimum at $\boldsymbol{a} \iff \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a}) > 0 \iff \frac{\partial^2 f}{\partial y^2}(\boldsymbol{a}) > 0$.*
  (ii) *f has a local maximum at $\boldsymbol{a} \iff \frac{\partial^2 f}{\partial x^2}(\boldsymbol{a}) < 0 \iff \frac{\partial^2 f}{\partial y^2}(\boldsymbol{a}) < 0$.*

(b) *f has a saddle point at $\boldsymbol{a}$ if and only if $D < 0$.*

Proof. This is just a special case of the theorem, together with the fact that $A = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ is positive definite (repsectively negative definite) if and only if $\det A > 0$ and $a > 0$ (respectively $a < 0$). See Theorem 1.2.1 and §§1.2.2 of Lecture15.

[1] If $g\colon (-\epsilon, \epsilon) \to \mathbf{R}$ is twice differentiable with a local minimum at 0, say $g(0) \geq g(t)$ for all $|t| < \delta$, then by the mean value theorem, for every $t \in (-\delta, 0)$, there exists $\xi = \xi(t) \in (t, 0)$ such that $g'(\xi) \leq 0$. Similarly, for every $t \in (0, \delta)$, there exists $\xi = \xi(t) \in (0, t)$ such that and $g'(\xi) \geq 0$. It follows that $\frac{g'(\xi(t)) - g'(0)}{\xi(t)} = \frac{g'(\xi(t))}{\xi(t)} \geq 0$ for all $0 < |t| < \delta$, whence $g''(0) \geq 0$.

1.2. **The special case of degree two polynomials.** Consider a degree two polynomial Φ in n-variables over $\mathbf{R}$. We studied these in §§2.2 of Lecture 15. Subsection 2.2.3 of *loc.cit.* showed us that a change of coordinates (via a translation and a "rotation") $\boldsymbol{x} \rightsquigarrow \boldsymbol{y}$ transforms $\Phi(\boldsymbol{x})$ to $\Phi^*(\boldsymbol{y}) = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2 + \varrho = 0$. A little thought shows that if all the $\lambda_i \neq 0$ then there is a unique critical point (in the new coordinates this is $\mathbf{0}$, but one has to transform back to the $\boldsymbol{x}$ coordinate system to get the original coordinates). Let us relate some of these ideas to Theorem 1.1.4.

We know from the discussion in Lecture 15 that there is a unique symmetric $n \times n$ matrix $A = [a_{ij}]$ and a linear form $P(\boldsymbol{x}) = p_1x_1 + \cdots + p_nx_n$ such that

$$\Phi(\boldsymbol{x}) = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1\le i<j\le n} 2a_{ij}x_ix_j + \sum_{i=1}^{n} p_ix_i + r = \boldsymbol{x}^tA\boldsymbol{x} + P(\boldsymbol{x}) + r \qquad (\boldsymbol{x} \in \mathbf{R}^n).$$

Since Φ is of degree two, the Hessian matrix function, $H(f) = [D_{ij}\Phi]$, is a constant matrix, which we denote H. A simple computation shows that $D_iD_j\Phi = 2a_{ij}$ for $i \le j$, and hence (since $D_{ij}\Phi = D_{ji}\Phi$ and $a_{ij} = a_{ji}$) for all $1 \le i, j \le n$. In other words

$$H = 2A. \tag{1.2.1}$$

Thus H is non-singular (respectively, positive definite, respectively negative definite) if and only if A is non-singular (respectively, positive definite, respectively negative definite). The point is, H can be read off from the homogenous degree two part of Φ without computations, since A can be so read off. Moreover, we note that

$$D_i\Phi = 2\sum_{j=1}^{n} a_{ij}x_j + p_i$$

giving

$$\boldsymbol{\nabla}\Phi(\boldsymbol{x}) = \Big(2\sum_{j=1}^{n} a_{1j}x_j + p_1, 2\sum_{j=1}^{n} a_{2j}x_j + p_2, \ldots, 2\sum_{j=1}^{n} a_{nj}x_j + p_n\Big). \tag{1.2.2}$$

Since $(\boldsymbol{\nabla}\Phi)^t = \Phi'$, (1.2.2) shows that the critical points of Φ are the solutions of the system of equations

$$2\sum_{j=1}^{n} a_{ij}x_j + p_i = 0 \qquad (i = 1, \ldots, n).$$

In other words, if $\boldsymbol{p} = (p_1, \ldots, p_n)$, the critical points of Φ are the solutions of

$$H\boldsymbol{x} = -\boldsymbol{p}. \tag{1.2.3}$$

Now suppose H is non-singular. Then (1.2.3) has a unique solution, i.e. Φ has exactly one critical point. Call it $\boldsymbol{a}$. Theorem 1.1.4 tells us that this is a saddle point if H, or equivalently A, is indefinite, it is a minimum if H (or A) is positive definite, and a maximum if H (or A) is negative definite. The last two statements can also be deduced by noting that after a change of coordinates $\boldsymbol{x} \rightsquigarrow \boldsymbol{y}$ involving a translation and a orthogonal transformation,[2] $\Phi(\boldsymbol{x})$ in new coordinates looks like $\Phi^*(\boldsymbol{y}) = \lambda_1y_1^2 + \cdots + \lambda_ny_n^2 + \varrho$ where the λ_i are the eigenvalues of A. If A is positive definite, the λ_i are all positive, and clearly $\sum_{i=1}^{n} \lambda_iy_i^2 + \varrho > \varrho$ for all $\boldsymbol{y} \neq \mathbf{0}$. Thus

[2]Neither operation changes the shape of the graph of Φ

Φ^* has a unique point of minimum, namely $\mathbf{0}$, and it follows that Φ has a unique minimum too. A similar analysis can be made when H is negative definite.

Examples 1.2.4. Let us put the above theoretical discussions to use in some examples.

1. Let

$$f(x,y) = 2x^2 + 3y^2 - 4xy - 12x + 14y + 21.$$

This is a degree two polynomial, and homogenous degree two summand is the quadratic form $Q(x,y) = 2x^2 + 3y^2 - 4xy$. The associated symmetric matrix is $A = \left[\begin{smallmatrix} 2 & -2 \\ -2 & 3 \end{smallmatrix}\right]$.[3] It follows that $H = \left[\begin{smallmatrix} 4 & -4 \\ -4 & 6 \end{smallmatrix}\right]$. Now $\det A = 2 > 0$, and the $(1,1)^{\text{th}}$ entry of A is 2 which is positive. Thus A, and hence H, is positive definite. The vector $\boldsymbol{p}$ in (1.2.3) is given by the coefficients of x and y, and hence $\boldsymbol{p} = (-12, 14)$.

Since H is non-singular, f is a unique critical point which is the solution of

$$\begin{bmatrix} 4 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -14 \end{bmatrix}.$$

The solution is $(x,y) = (2,-1)$. Since H is positive definite, this critical point of f is a minimum for f. Now $f(2,-1) = 2(2^2) + 3(-1)^2 - 4(2)(-1) - 12(2) + 14(-1) + 21 = 2$ and hence the minimum value of f is 2.

Since 2 is the minimum value of f, the level set $f(x,y) = c$ is empty when $c < -2$. It is exactly one point when $c = 2$, namely the point $(2,-1)$. It is an ellipse (because A is positive definite) when $c > 2$. The graph of $f(x,y)$, i.e. the locus of $z = f(x,y)$, is a paraboloid, with cross sections (level curves) being ellipses for levels above 2, and with vertex at $(2,-1,2)$.

2. Let f be as in Example **1.** It is not hard to see that

$$f(x,y) = 2(x-2)^2 + 3(y+1)^2 - 4(x-2)(y+1) + 2.$$

So if $x^\flat = x - 2$ and $y^\flat = y + 1$ (this is the change of coordinates when one shifts the origin to $(2,-1)$), then the transformed quadric function is $f^\flat(x^\flat, y^\flat) = 2(x^\flat)^2 + 3(y^\flat)^2 - 4(x^\flat)(y^\flat) + 2$. As we observed, the matrix A in Example **1.** is positive definite. Therefore, by an orthogonal transform, we can get a new system of co-ordinates (x^*, y^*) such that $f^\flat$ in the new coordinates is $f^*(x^*, y^*) = \lambda_1(x^*)^2 + \lambda_2(y^*)^2 + 2$ where λ_1 and λ_2 are eigenvalues of A. Since A is positive definite, these eigenvalues are positive. This means $f^*(x^*, y^*) > 2$ if $(x^*, y^*) \neq \mathbf{0}$. This is another way, without calculus, of seeing that 2 is the minimum value of f. The technique does not work so well when f is not a quadratic polynomial.

3. Let

$$f(x,y,z) = 14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy + 18x - 18y + 5 = 0.$$

The associated quadratic form is

$$Q(x,y,z) = 14x^2 + 14y^2 + 8z^2 - 4yz - 4zx - 8xy.$$

[3]The diagonal entries are the coefficients of x^2 and y^2 and the off diagonal entries are half the coefficient of xy.

The associated matrix is

$$A = \begin{bmatrix} 14 & -4 & -2 \\ -4 & 14 & -2 \\ -2 & -2 & 8 \end{bmatrix}.$$

It is not hard to compute the eigenvalues. They are $\lambda_1 = 6$, $\lambda_2 = 12$, $\lambda_3 = 18$. It follows that A is positive definite. The Hessian is the constant matrix

$$H = \begin{bmatrix} 28 & -8 & -4 \\ -8 & 28 & -4 \\ -4 & -4 & 16 \end{bmatrix}.$$

Since $H = 2A$, this too is positive definite. Thus there is only one critical point $\boldsymbol{a}$ and it is a point of minimum of f. This critical point is a solution of (1.2.3), i.e. $\boldsymbol{a}$ is the unique solution of:

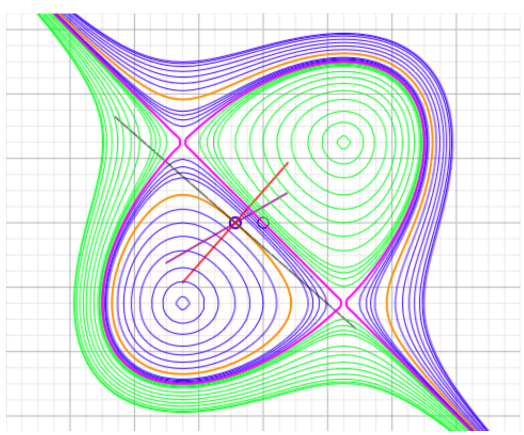
$$\begin{bmatrix} 28 & -8 & -4 \\ -8 & 28 & -4 \\ -4 & -4 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 18 \\ 0 \end{bmatrix}$$

I leave it to you to solve this and find $\boldsymbol{a}$. The value of f at $\boldsymbol{a}$, i.e. $f(\boldsymbol{a})$, is the minimum value of f.

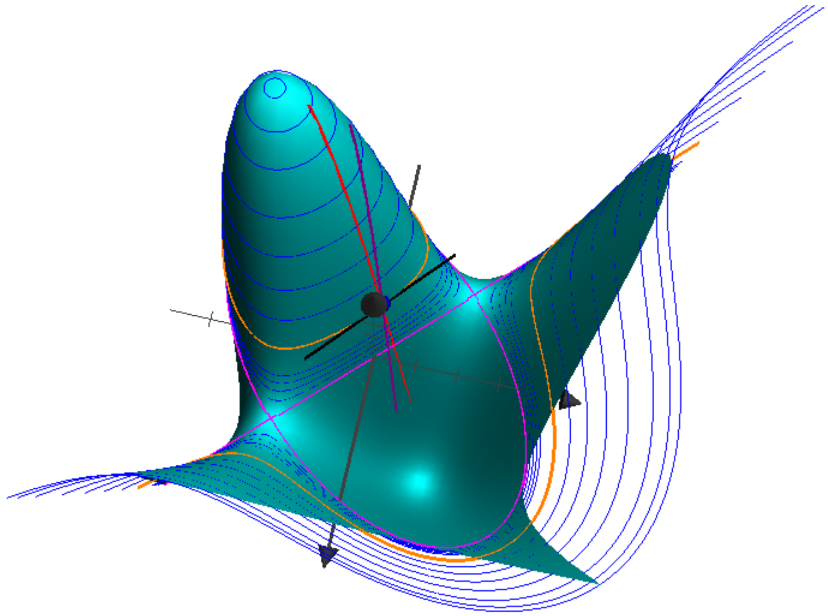
4. Let

$$f(x, y) = x^3 + y^3 - 3x - 3y.$$

This is Example 1.1.4 of Lecture 11. The contours are (see Lecture 11 again):



From the contours, they seem to be two extreme points (the centres of the closed curves) and two saddle points (look carefully at the complicated pink curve). Here is a picture of the graph of f (a different view from those given in Lecture 11).



A peak, a trough (the lowest point) of a valley, and two saddle points are clearly seen. To work out this example, we cannot use the tricks used earlier because f is not of degree two. Here are the computations:

$$\nabla f(x, y) = (3x^2 - 3, 3y^2 - 3).$$

The critical points are therefore the solutions of the system of equations

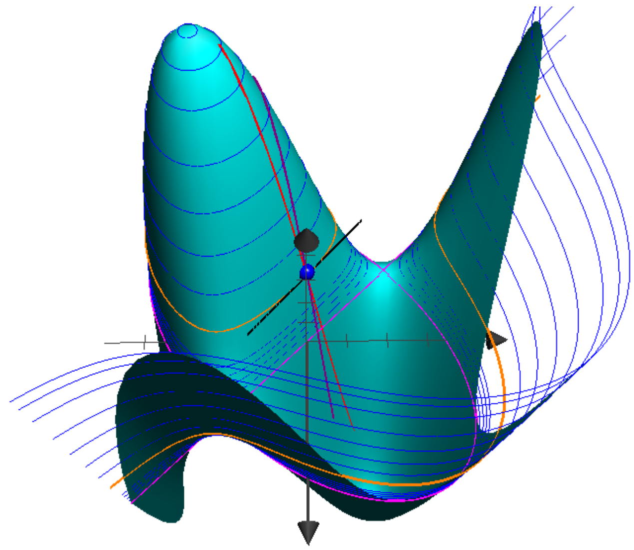
$$x^2 - 1 = 0 \quad \text{and} \quad y^2 - 1 = 0$$

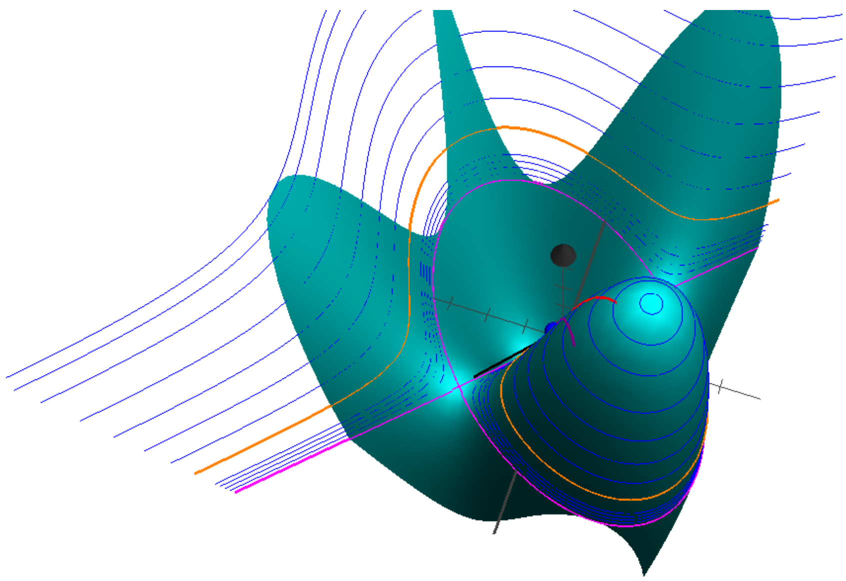
It follows that the critical points are $\boldsymbol{a}_1 = (1, 1)$, $\boldsymbol{a}_2 = (-1, 1)$, $\boldsymbol{a}_3 = (-1, -1)$, and $\boldsymbol{a}_4 = (1, -1)$. To work out the maxima, minima, and saddle points we need to compute the Hessian. However, to shore up our intuition, look at the contours (level curves) we have above the picture of the surface. It is clear that at $\boldsymbol{a}_2 = (-1, 1)$ and $\boldsymbol{a}_4 = (1, -1)$ we have saddles (the level curves cross each other at each of these points). If you can't see that from the contours, look at the surface. To return to calculations, and writing $H(x, y)$ for $H(f)(x, y)$, we have

$$H(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}.$$

Note that unlike the quadratic case, this is not a constant matrix. It depends upon (x, y). At the four critical points the Hessian is $H(\boldsymbol{a}_1) = \left[\begin{smallmatrix} 6 & 0 \\ 0 & 6 \end{smallmatrix}\right]$, $H(\boldsymbol{a}_2) = \left[\begin{smallmatrix} -6 & 0 \\ 0 & 6 \end{smallmatrix}\right]$, $H(\boldsymbol{a}_3) = \left[\begin{smallmatrix} -6 & 0 \\ 0 & -6 \end{smallmatrix}\right]$, and $H(\boldsymbol{a}_4) = \left[\begin{smallmatrix} 6 & 0 \\ 0 & -6 \end{smallmatrix}\right]$. This means $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$ are saddle points for f, as we predicted looking at various pictures. The point $\boldsymbol{a}_1$ is a point of relative minimum for f, and $\boldsymbol{a}_3$ is a point of relative maximum for f.

Here are two more pictures of the surface (why not?):





About these notes. This lecture was supposed to be given on March 25, 2020. Classes got suspended on March 17 because of the coronavirus COVID 19 pandemic, and all teaching moved online. These course notes are a reasonably faithful record of the lectures given (before the shutdown) at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.

References

[R] W. Rudin, *Principles of Mathematical Analysis*, (Third Edition), McGraw-Hill, New Delhi, 1976.