# LECTURE 15 SUPPLEMENT

**Date of Lecture:** March 18, 2020

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An $n$-tuple $(x_1, \ldots, x_n)$ of symbols ($x_i$ not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set $S$ to a product set $T_1 \times \cdots \times T_n$ will often be written as an $n$-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with $f_i$ a map from $S$ to $T_i$, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \; \|_2$ and we will simply denote it as $\| \; \|$.

Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

## 1. Inner Products and Positive Definite Operators

As before, let $V$ be a finite dimensional inner product space over $\mathbf{K}$ with inner product $\langle \;, \; \rangle$.

1.1. Let $T\colon V \to V$ be positive definite. Set $\langle v, w\rangle_T = \langle Tv, w\rangle$ for $v, w \in V$. It is clear that $\langle \;\; \rangle_T \colon V \times V \to \mathbf{K}$ is linear in the first argument, conjugate linear in the second argument, $\langle v, v\rangle_T \geq 0$ for all $v \in V$, and $\langle v, v\rangle_T = 0$ if and only if $v = 0$. Moreover, $\langle v, w\rangle_T = \langle Tv, w\rangle = \overline{\langle w, Tv\rangle} = \overline{\langle Tw, v\rangle}$ since $T$ is self-adjoint. In other words,

$$\langle v, w\rangle_T = \overline{\langle w, v\rangle}_T.$$

Thus $\langle \;, \; \rangle_T$ is a (possible) second inner product on $V$.

Conversely suppose $\langle\langle \;, \; \rangle\rangle$ is a second inner product on $V$. Pick an orthonormal basis for $f_1, \ldots, f_n$ of $V$ for the original inner product $\langle \;, \; \rangle$. Set

$$a_{ij} = \langle\langle f_j, f_i\rangle\rangle \qquad\qquad (i, j = 1, \ldots, n).$$

Define a $\mathbf{K}$-linear map $T\colon V \to V$ by the formula

$$T(f_j) = \sum_{i=1}^{n} a_{ij} f_i.$$

The matrix of $T$ with respect to $f_1, \dots, f_n$ is clearly $A = [a_{ij}]$. Then

$$\begin{aligned}\langle T(\sum_{j=1}^n x_j f_j), \sum_{i=1}^n y_i f_i\rangle &= \langle \sum_{j=1}^n x_j \sum_{k=1}^n a_{kj} f_k, \sum_{i=1}^n y_i f_i\rangle \\ &= \sum_{j=1}^n \sum_{k=1}^n \sum_{i=1}^n a_{kj} x_j \bar{y}_i \langle f_k, f_i\rangle \\ &= \sum_{j=1}^n \sum_{i=1}^n a_{ij} x_j \bar{y}_i \\ &= \sum_{j=1}^n \sum_{i=1}^n x_j \bar{y}_i \langle\langle f_j, f_i\rangle\rangle \\ &= \langle\langle \sum_{j=1}^n x_j f_j, \sum_{i=1}^n y_i f_i\rangle\rangle.\end{aligned}$$

Thus $\langle\ ,\ \rangle_T = \langle\langle\ ,\ \rangle\rangle$.

**Theorem 1.1.1.** *There is a one-to-one correspondence between positive definite operators on $V$ and inner products on $V$, namely the corespondence given by $T \mapsto \langle\ ,\ \rangle_T$.*

*Proof.* From the discussion preceding the statement, we only have to show that if $\langle\ ,\ \rangle_T = \langle\ ,\ \rangle_S$ then $T = S$. Pick an orthonormal basis $f_1, \dots, f_n$ of $V$ and suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are the respective matrices of $T$ and $S$ with respect to this basis. Then $a_{ij} = \langle T f_j, f_i\rangle = \langle f_j, f_i\rangle_T = \langle f_j, f_i\rangle_S = \langle S f_j, f_i\rangle = b_{ij}$. Thus $A = B$, and hence $T = S$. □

## 2. **Quadratic forms**

We fix a finite dimensional inner product space $V$ over $\mathbf{R}$

2.1. **The classical approach over R.** Classically a quadratic form in two variables[1] is

$$Q(x, y) = Ax^2 + By^2 + 2Cxy \tag{2.1.1}$$

where $A, B, C \in \mathbf{R}$ and $x, y$ are variables. In other words $Q(x, y) \in \mathbf{R}[x, y]$ is a homogeneous polynomial of degree 2. Such forms are intimately associated to ellipses and hyperbolas. In fact $Q(x, y) = 0$ is the equation of an ellipse or a hyperbola with "centre at $\mathbf{0}$" and vice versa.

In three variables quadratic forms are classically defined as the following kinds of forms in $\mathbf{R}[x, y, z]$, with $A, B, C, D, E, F \in \mathbf{R}$.

$$Q(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz \tag{2.1.2}$$

In other words a quadratic form in three variables is again a homogeneous degree two polynomial. The equation $Q(x, y, z) = 0$ is associated with ellipsoids and hyperboloids (of one sheet and of two sheets). These are respectively of the form $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} A & C \\ C & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. One can define quadratic forms in the

[1]One can define quadratic forms over any field. This is a rich field of research connected intimately with Azumaya algebras, central simple algebras, and advanced forms of Galois Theory having to do with étale topology.

same way in many variables. Note that the matrices we wrote down are symmetric, and such matrices are diagonalisable over $\mathbf{R}$ via a orthogonal matrix.

2.2. **Quadratic forms in a co-ordinate free way.** A map $Q\colon V \to \mathbf{R}$ is said to be a *quadratic form* if there exists a *self-adjoint* operator $T\colon V \to V$ such that

$$Q(v) = \langle Tv, v\rangle \qquad (v \in V).$$

We write $Q = Q_T$ if we wish to emphasise the role of $T$.

Let $Q = Q_T$ be a quadratic form on $V$. Let $f_1, \dots, f_n$ be an orthonormal basis of $V$. Let $A$ be the matrix of $T$ with respect to $\mathscr{B} = \{f_i\}$ and $a_{ij}$ be the $(i,j)^{\text{th}}$ entry of $A$. Recall that one can compute $a_{ij}$ by the formula $a_{ij} = \langle Tf_j, f_i\rangle$. It follows that

$$\begin{aligned} Q(x_1f_1 + \cdots + f_nx_n) &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \\ &= \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1\le i<j\le n} 2a_{ij}x_ix_j \end{aligned} \tag{2.2.1}$$

for $(x_1, \dots, x_n) \in \mathbf{R}^n$. We are using the fact that $A$ is symmetric and hence $a_{ij} = a_{ji}$ to get the coefficient 2 in the second summand of the last line. Thus the entries of the $A$ are completely determined by the coefficients of the homogeneous quadratic polynomial (2.2.1) and vice versa. Hence if $Q_T = Q_S$ then $T = S$.

**2.2.2.** A quadratic form on $\mathbf{R}^n$ is simply a homogeneous degree two polynomial in $n$-variables, i.e. any polynomial $Q(x_1, \dots, x_n) \in \mathbf{R}[x_1, \dots, x_n]$ of the form

$$Q(\boldsymbol{x}) = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1\le i<j\le n} 2a_{ij}x_ix_j,$$

where by convention we write the coefficient of $x_ix_j$ for $i < j$ as $2a_{ij}$. The convention has the advantage that if $A$ is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $a_{ij}$ when $i \le j$ and $a_{ji}$ when $i > j$ , then

$$Q(\boldsymbol{x}) = \boldsymbol{x}^t A\boldsymbol{x}. \tag{2.2.2.1}$$

**2.2.3.** We know that $V$ has an orthonormal basis consisting of eigenvectors of $T$. Let $\mathscr{B} = \{f_i\}$ be such an orthonormal basis, with $\lambda_i \in \mathbf{R}$, $i = 1, \dots, n$ the eigenvalues corresponding to the eigenvectors $f_1, \dots, f_n$. Then clearly $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and hence,

$$Q(x_1f_1 + \cdots + f_nx_n) = \lambda_1x_1^2 + \ldots + \lambda_nx_n^2 \qquad ((x_1, \dots, x_n) \in \mathbf{R}^n). \tag{2.2.3.1}$$

**About these notes.** These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.