LECTURE 15

Date of Lecture: March 18, 2020

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \ldots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$.

☡ Note that $(x_1, \ldots, x_n) \neq [x_1 \ldots x_n]$. Each side is the transpose of the other.

1. Positive definite and negative definite operators

We fix a finite dimensional inner product space V over $\mathbf{K}$

1.1. The main definitions.

Inner products on V are in some sense parameterised by self-adjoint operators which have the following property.

Theorem 1.1.1. *Let $T\colon V \to V$ be a self-adjoint operator. The following are equivalent.*

(a) $\langle Tv, v\rangle > 0$ *for all* $v \in V \smallsetminus \{0\}$ *(respectively* $\langle Tv, v\rangle < 0$ *for all* $v \in V \smallsetminus \{0\}$*).*
(b) *All the eigenvalues of T are positive (respectively negative).*

Proof. Since T is self-adjoint all its eigenvalues are real and V has an orthonormal basis, say $\mathscr{B} = \{f_1, \ldots, f_n\}$, consisting of eigenvectors of T (see Theorems 2.2.5 and 2.3.2 of of Lecture 13). Let $\lambda_i \in \mathbf{R}$ be the eigenvalue associated with f_i, $i = 1, \ldots, n$. Recall, this means $Tf_i = \lambda_i f_i$, $i = 1, \ldots, n$.

Suppose $\langle Tv, v\rangle > 0$ for all non-zero v in V. Then $\lambda_i = \langle \lambda_i f_i, f_i\rangle = \langle Tf_i, f_i\rangle > 0$ for $i = 1, \ldots n$. Similarly, if $\langle Tv, v\rangle < 0$ for all non-zero v in V, then all the eigenvalues of T are negative.

Conversely, suppose all the eigenvalues of T are positive. Let $v \in V \setminus \{0\}$. There are unique scalars $x_i \in \mathbf{K}$, $i = 1, \ldots, n$ such that $v = \sum_{i=1}^{n} x_i f_i$, and since $v \neq 0$, at least one $x_i \neq 0$, say x_{i_0}. Now,

$$\langle Tv, v\rangle = \langle T(\sum_{=i}^{n} x_i f_i), \sum_{i=1}^{n} x_i f_i\rangle = \sum_{i=1}^{n} \lambda_i |x_i|^2 \geq \lambda_{i_0} |x_{i_0}|^2 > 0.$$

Similarly, if all the eigenvalues of T are negative then $\langle Tv, v\rangle < 0$ for all non zero v in V. □

Definition 1.1.2. Let $T\colon V \to V$ be a self-adjoint operator. T is said to be *positive definite* (respectively *negative definite*) if $\langle Tv, v\rangle > 0$ for all $v \in V \setminus \{0\}$ (respectively if $\langle Tv, v\rangle < 0$ for all $v \in V \setminus \{0\}$). T is said to be *positive semi-definite* (respectively *negative semi-definite*) if $\langle Tv, v\rangle \geq 0$ for all $v \in V \setminus \{0\}$ (respectively if $\langle Tv, v\rangle \leq 0$ for all $v \in V \setminus \{0\}$). If T is neither positive nor negative semi-definite it is called *indefinite*

Corollary 1.1.3. *If T is positive definite* $\det T$ *is positive, and if T is negative definite, then* $(-1)^{\dim V} \det T$ *is positive.*

Proof. This is obvious since the determinant of a linear operator is the product of its eigenvalues (counted with multiplicity). □

Proposition 1.1.4. *Let $T\colon V \to V$ be a self-adjoint operator and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. T is positive semi-definite (respectively negative semi-definite) if and only if $\lambda_i \geq 0$ (respectively $\lambda_i \leq 0$) for every $i = 1, \ldots, n$.*

Proof. The proof is *mutatis mutandis* the proof of Theorem 1.1.1. □

Definition 1.1.5. Let H be a hermitian $n \times n$ matrix. H is said to be *positive (respectively negative) definite* if $\bar{\boldsymbol{x}}^t H \boldsymbol{x} > 0$ (respectively $\bar{\boldsymbol{x}}^t H \boldsymbol{x} < 0$) for every non-zero $\boldsymbol{x}$ in $\mathbf{C}^n$. It is *positive (respectively negative) semi-definite* if $\bar{\boldsymbol{x}}^t H \boldsymbol{x} \geq 0$ (respectively $\bar{\boldsymbol{x}}^t H \boldsymbol{x} \leq 0$). H is *indefinite* if it is neither positive nor negative semi-definite.

Remark 1.1.6. Let H be a hermitian matrix. It is clear$H$ is positive or negative definite if and only if the linear operator $\boldsymbol{x} \mapsto H\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbf{C}^n$ is positive or negative definite respectively. Indeed

$$\langle H\boldsymbol{x}, \boldsymbol{x}\rangle = \bar{\boldsymbol{x}}^t (H\boldsymbol{x})$$

giving, quite easily, the assertion. In particular H is positive definite if and only if all its eigenvalues are positive and it is negative definite if and only if all its eigenvalues are negative. Similar statements can be made for the semi-definite cases.

Note that if H as above is symmetric (i.e. if all its entries are real), then H is positive definite if and only if the operator $\boldsymbol{x} \mapsto H\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbf{R}^n$ is positive definite on $\mathbf{R}^n$. This follows easily from the characterisation of positive definiteness of operators via eigenvalues. Other analogous statements for H negative definite, or one of the indefinite types, also hold, for similar reasons.

1.2. **Positive definite 2x2 real matrices.** We begin with the following observation. Suppose $\mathbf{K} = \mathbf{R}$, $V = \mathbf{R}^n$ and $T\colon V \to V$ is positive definite. Let $A = [a_{ij}]$ be the matrix of T with respect to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then the diagonal entries a_{ii} are all positive. Indeed $a_{ii} = \langle T\mathbf{e}_i, \mathbf{e}_i\rangle > 0$, $i = 1, \ldots, n$. There is a partial converse when $n = 2$ useful in many maxima and minima problems.

Proposition 1.2.1. *Let* $A = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ *be a symmetric* 2×2 *matrix. Then* A *is positive definite if and only if* $\det A > 0$ *and* $a > 0$*. It is negative definite if and only if* $\det A > 0$ *and* $a < 0$.

Proof. Let $\boldsymbol{f}_1 = (\gamma_{11}, \gamma_{21})$ and $\boldsymbol{f}_2 = (\gamma_{12}, \gamma_{22})$ be eigenvectors of A in $\mathbf{R}^2$ with $\boldsymbol{f}_1, \boldsymbol{f}_2$ forming an orthonormal basis of $\mathbf{R}^2$. Let λ_1 and λ_2 be the eigenvalues of A associated with $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ respectively. We know $\lambda_1, \lambda_2 \in \mathbf{R}$ and $\det A = \lambda_1\lambda_2$.

Let $\Gamma = \left[\begin{smallmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{smallmatrix}\right]$. We know that $\Gamma^t A \Gamma = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ and that $\Gamma^{-1} = \Gamma^t$ (see §§1.2 of Lecture 14). It follows that $A = \Gamma\left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]\Gamma^t$. Since $a = \mathbf{e}_1^t A \mathbf{e}_1 = \mathbf{e}_1^t \Gamma \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]\Gamma^t \mathbf{e}_1 = \left[\begin{smallmatrix} \gamma_{11} & \gamma_{12} \end{smallmatrix}\right]\left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} \gamma_{11} \\ \gamma_{12} \end{smallmatrix}\right]$, this yields

$$(*) \qquad a = \lambda_1\gamma_{11}^2 + \lambda_2\gamma_{12}^2.$$

If A is positive definite, we have already observed that the diagonal entries are positive. In particular $a > 0$. Moreover according to Corollary 1.1.1 $\det A$ is positive. Suppose conversely that $a > 0$ and $\det A > 0$. From $(*)$ it is clear that at least one of λ_1 and λ_2 is positive. Now $\lambda_1\lambda_2 = \det A > 0$, and hence λ_1 and λ_2 have the same sign. Thus λ_1 and λ_2 are both positive. This means A is positive definite.

The negative definite case is proved in a similar way, or simply by observing that A is negative definite if and only if $-A$ is positive definite. □

1.2.2. Let A be as in the theorem. The matrix of $\boldsymbol{x} \mapsto A\boldsymbol{x}$ with respect to the ordered basis $\{\mathbf{e}_2, \mathbf{e}_1\}$ is $C = \left[\begin{smallmatrix} c & b \\ b & a \end{smallmatrix}\right]$ and C is positive definite (or negative definite) if and only if A is. It follows that A is positive definite if and only if $\det A > 0$ and $c > 0$. Further, A is negative definite if and only if $\det A > 0$ and $c < 0$. Each of these statements can of course be proved directly as in the proof of the theorem.

2. Quadratic forms

2.1. **Classical approach.** Classically a quadratic form in two variables (over any field, not necessarily $\mathbf{R}$ or $\mathbf{C}$, but we restrict ourselves to $\mathbf{R}$) is a form of the following kind

$$(2.1.1) \qquad Q(x, y) = Ax^2 + By^2 + 2Cxy$$

where $A, B, C \in \mathbf{R}$ and x, y are variables. In other words $Q(x, y) \in \mathbf{R}[x, y]$ is a homogeneous polynomial of degree 2. Such forms are intimately associated to ellipses and hyperbolas. In fact $Q(x, y) = 0$ is the equation of an ellipse or a hyperbola with "centre at $\mathbf{0}$" and vice versa.

In three variables quadratic forms are classically defined as the following kinds of forms in $\mathbf{R}[x, y, z]$, with $A, B, C, D, E, F \in \mathbf{R}$.

$$(2.1.2) \qquad Q(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz$$

In other words a quadratic form in three variables is again a homogeneous degree two polynomial. The equation $Q(x, y, z) = 0$ is associated with ellipsoids and hyperboloids (of one sheet and of two sheets). These are respectively of the form $\left[\begin{smallmatrix} x & y \end{smallmatrix}\right]\begin{bmatrix} A & C \\ C & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\left[\begin{smallmatrix} x & y & z \end{smallmatrix}\right]\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. One can define quadratic forms in the same way in many variables. Note that the matrices we wrote down are symmetric, and such matrices are diagonalisable over $\mathbf{R}$ via an orthogonal matrix.

More generally, in n-variables, a quadratic form is just a homogeneous polynomial of degree two in n-variables. Thus a typical quadratic form looks like

$$Q(\boldsymbol{x}) = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1\le i<j\le n} 2a_{ij}x_ix_j \tag{2.1.3}$$

where by convention one writes the coefficient of x_ix_j for $i<j$ as $2a_{ij}$. Set $a_{ij}=a_{ji}$ when $i>j$ and let $A=[a_{ij}]$, the $n\times n$ *symmetric* matrix whose $(i,j)^{\text{th}}$ entry is a_{ij}. Then

$$Q(\boldsymbol{x}) = \boldsymbol{x}^t A\boldsymbol{x} \tag{2.1.4}$$

Note that Q determines A and vice-versa. We will often write Q_A for Q when we wish to emphasise that Q arises from the symmetric matrix A.

2.2. **Quadric hypersurfaces.** A *quadric hypersurface* S in $\mathbf{R}^n$ (or more precisely an *affine quadric hypersurface in affine n-space*) is the locus of points satisfying an equation of the form

$$\Phi(x_1,\dots,x_n)=0 \tag{2.2.1}$$

where $\Phi\in\mathbf{R}[T_1,\dots,T_n]$ is a polynomial of degree 2.

Examples 2.2.2. Here we examine the simplest examples, namely quadric hypersurfaces of the form

$$\lambda_1x_1^2+\ldots+\lambda_nx_n^2=c \tag{2.2.2.1}$$

where the λ_i and c are constants. Equivalently,

$$\begin{bmatrix}x_1 & x_2 & \dots & x_n\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=c. \tag{2.2.2.2}$$

This is a quadric surface as seen by setting $\Phi=Q-c$ where $Q(\boldsymbol{x})=\lambda_1x_1^2+\ldots+\lambda_nx_n^2$. In what follows

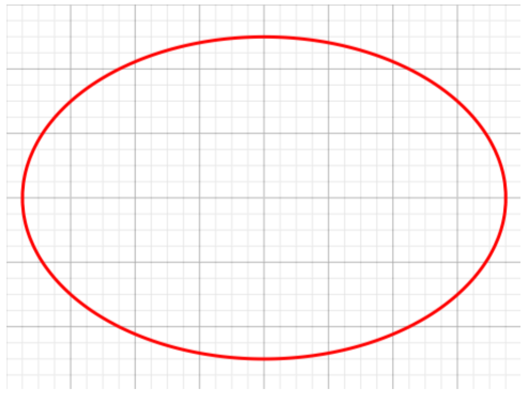
$$A := \operatorname{diag}(\lambda_1,\dots,\lambda_n).$$

1. *A* **positive definite and** $n=2$**.** These are curves of the form $\lambda_1x^2+\lambda_2y^2=c$, wiith λ_1 and λ_2 positive. If $c<0$, then the locus of points satisfying the equation is empty. If $c=0$ then only the origin satisfies the equation. If $c>0$ we have ellipses. The *standard form* of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a,b>0$, and in this case the two *principal axes* of the ellipse are the line segments from $(-a,0)$ to $(a,0)$ and from $(0,-b)$ to $(0,b)$. The larger of the axes is called the *major axis* and the smaller the *minor axis*. If they are both of the same length (i.e. $a=b$) then the ellipse is a circe of radius a centred at $\mathbf{0}$.

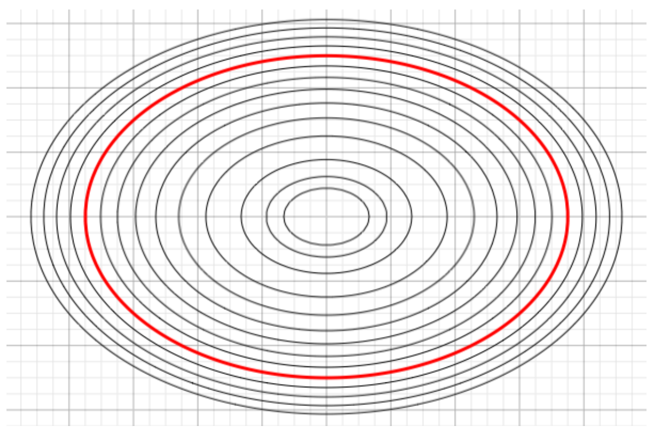
   Consider

   $$\frac{x^2}{9}+\frac{y^2}{4}=1.$$

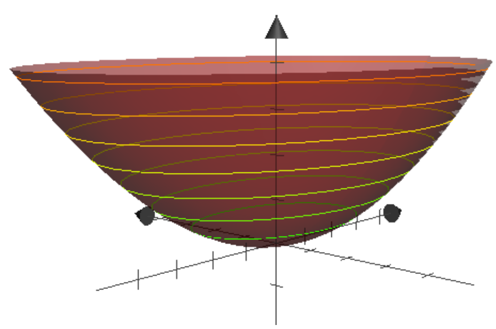
   Its graph is below. The "centre" is $\mathbf{0}$. The major axis is along the x-axis and is of length $2a=6$ and the minor axis is of length $2b=4$.



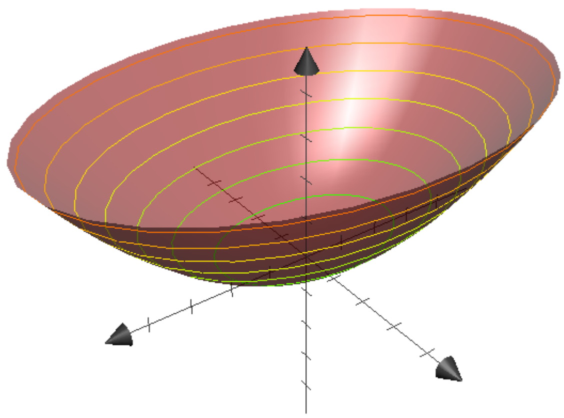
Let $Q(x, y) = \frac{x^2}{9} + \frac{y^2}{4}$. The above is the level curve $Q(x, y) = c$ for $c = 1$. Here are some more level curves where we allow c to vary (the red ellipse remains the one for $c = 1$).



As we discussed in Lecture 11 these level curves, also known as contours, give an idea of the graph of $z = Q(x, y)$. For the record, here is the graph of $z = Q(x, y)$.

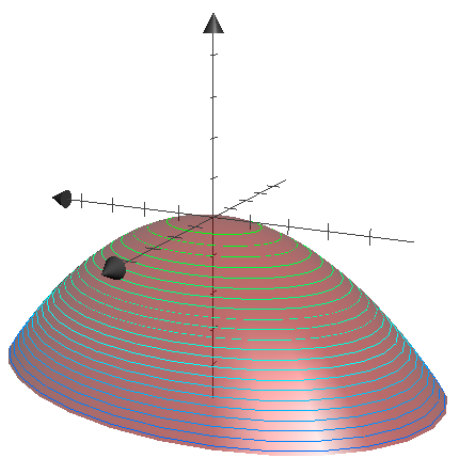


This is an *elliptic paraboloid*. You can see the contour curves on the surface. These cross-sections are ellipses. However cross sections with respect to any plane containing the z-axis is a parabola. Here is another angle of our elliptic paraboloid.

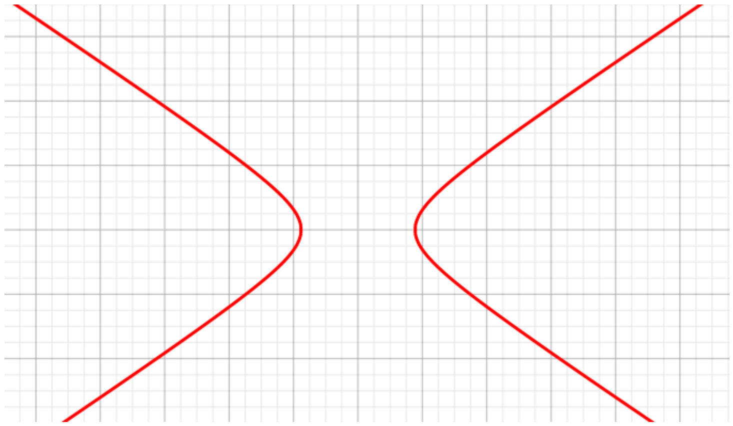


It is clear that Q has a minimum at $(x, y) = (0, 0)$. A little thought shows that the graph of any surface $z = \lambda_1 x^2 + \lambda_2 y^2$ with λ_1 and λ_2 positive must be a elliptic paraboloid turned upwards with minimum at $\mathbf{0}$.

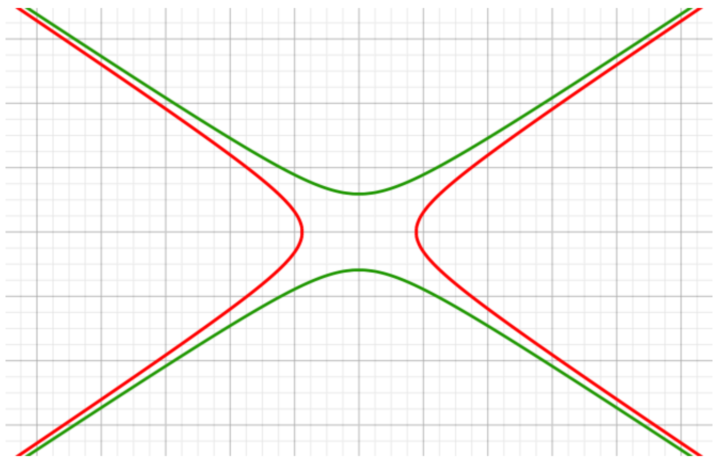
2. *A* **negative definite and** $n = 2$**.** Consider the case $Q(x, y) = \lambda_1 x^2 + \lambda_2 y^2$ where the λ_i are both negative. We have level curves $Q(x, y) = c$, These loci are non-empty only when $c \leq 0$, When $c = 0$ we have a point, otherwise we have exactly the ellipses we had before. The graph of $z = Q(x, y)$ ls again a elliptic paraboloid, but now turned downwards, with $\mathbf{0}$ being a point of maxima for Q (see below).



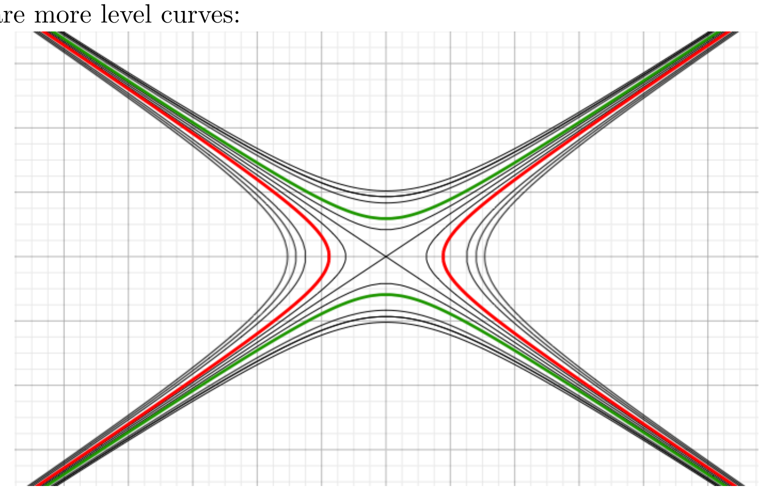
3. $\det A < 0$ **and** $n = 2$**.** Here $Q(x, y) = \lambda_1 x_1^2 + \lambda_2 x_2^2$ with $\lambda_1 \lambda_2 < 0$. The level curves $Q(x, y) = c$ are hyperbolas. Consider $Q(x, y) = 4x^2 - 9y^2$. Here is the level curve for $Q(x, y) = 2$



Here is are the level curves for $c = 2$ and $c = -2$

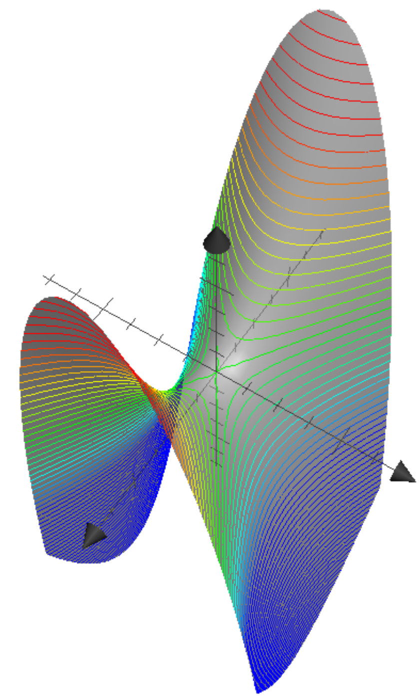


Here are more level curves:

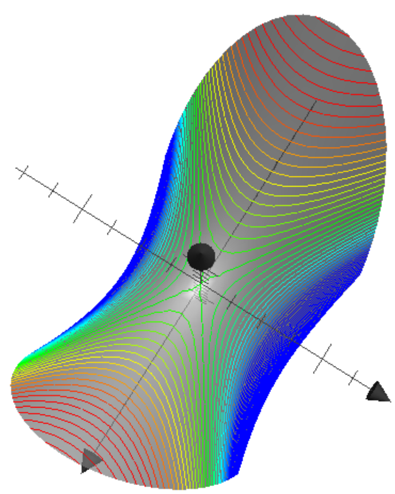


The hyperbolas opening upwards and downwards are for negative values of c. The hyperbolas opening to the right and to the left correspond to positive values of c. The two straight lines passing through the origin correspond to $c = 0$. This is regarded as a degenerate hyperbola. The equation is $4x^2 - 9y^2 = 0$, or, what is the same thing $(2x - 3y)(2x + 3y) = 0$ giving you the equation of a pair of lines.

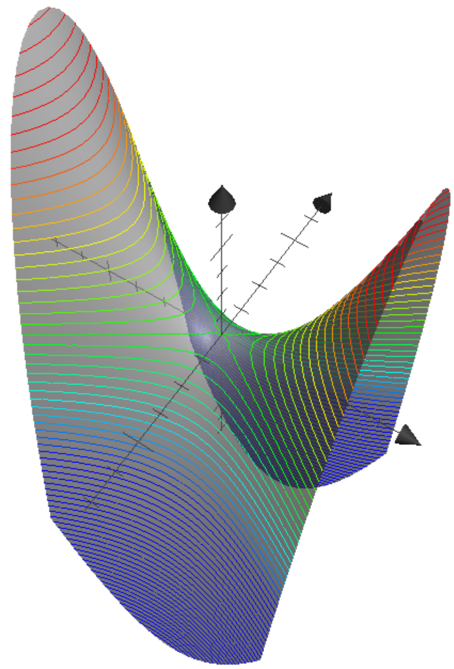
Here are three pictures of the surface $z = 4x^2 - 9y^2$ with contour lines on them.



The positive x-axis in the picture above is pointing to the left of the reader and downwards. The positive z-axis is pointing upwards as always.



The picture above is a "view from the top". The positive z-axis is pointing at the reader. The two other axes follow the usual right hand rule.



In this view, the positive x-axis is pointing to the right. The positive y-axis away from the reader, and the positive z-axis upwards.

Note that the origin is neither a minimum nor a maximum for Q. This is because (as we will see) A is neither positive definite not negative definite. The eigenspace corresponding to the positive eigenvalue 4 is the x-axis and the eigenspace corresponding to the negative eigenvalue -9 is the y-axis. Looking at the pictures it should be clear that travelling along the x-axis on the surface, you will reach a local minima at $\mathbf{0}$ and travelling along the y-axis on the surface, you will hit a local maxima at $\mathbf{0}$. This is a general phenomenon.

The point $\mathbf{0}$ is a *saddle point* of the surface S.

2.2.3. Changing co-ordinates to get a quadric equation into a "diagonal form". By changing co-ordinates via translating the origin and affecting an orthogonal transform, one can re-write the equation of S as the equation of a level surface of a particularly simple form, namely of the form in (2.2.2.1), i.e.

$$\lambda_1 x_1^2 + \ldots \lambda_n x_n^2 = c$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the symmetric matrix associated A associated with the homogeneous degree two summand of Φ, and c is a constant. This is seen as follows. First, if Φ is of the form

$$\Phi(x_1, \ldots, x_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 + p_1 x_1 + \cdots + p_n x_n + r$$

where the λ_i, p_i and r are constants, then one can complete the squares and re-write the the equation for S in the form

$$\sum_{i=1}^{n} \lambda_i (x_i - a_i)^2 + \varrho = 0$$

where $a_i = -\frac{p_2}{2\lambda_i}$ when $\lambda_i \neq 0$ (otherwise take it to be anything since the associated summand $\lambda_i(x_i - a_i)$ essentially does not appear in the sum). Thus a change of co-ordinates, namely *translating the origin* to $\boldsymbol{a} = (a_1, \dots, a_n)$ gives us the following equation for S

$$(*) \qquad \sum_{i=1}^{n} \lambda_i (x_i')^2 = c$$

where $x_i' = x_i - a_i$ are the new coordinates (with respect to the new origin) and $c = -\varrho$.

More generally

$$\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}) + P(\boldsymbol{x}) + r = \boldsymbol{x}^t A \boldsymbol{x} + P(\boldsymbol{x}) + r$$

where Q is a quadratic form, say $Q = Q_A$ with A symmetric, P is linear homogeneous, i.e. $P(\boldsymbol{x}) = \sum_{i=1}^n p_i x_i$, and r is a constant. Let Γ be an orthogonal matrix such that $\Gamma^t A \Gamma = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We know such an orthogonal Γ exists. Consider the change of coordinates $\boldsymbol{x} \rightsquigarrow \boldsymbol{x}^*$ given by

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{21} & \cdots & \gamma_{n1} \\ \gamma_{12} & \gamma_{22} & \cdots & \gamma_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1n} & \gamma_{2n} & \cdots & \gamma_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where of course γ_{ij} is the $(i,j)^{\text{th}}$ entry of Γ. In other words

$$(2.2.3) \qquad \boldsymbol{x}^* = \Gamma^t \boldsymbol{x} = \Gamma^{-1} \boldsymbol{x}.$$

Thus $Q(\boldsymbol{x}) = \boldsymbol{x}^t A \boldsymbol{x} = \boldsymbol{x}^{*t} \Gamma^t A \Gamma \boldsymbol{x}^* = \sum_{i=1}^n \lambda_i (x_i^*)^2$, and if $P^*(\boldsymbol{x}^*) = P(\Gamma \boldsymbol{x}^*)$, then S is given by

$$\sum_{i=1}^{n} \lambda_i (x_i^*)^2 + P^*(\boldsymbol{x}^*) + r = 0$$

Completing the square, as we did earlier, we can get to a system of co-ordinates $\boldsymbol{x}'$ (by translating the origin) so that the equation of S in the new system is of the form $(*)$ and hence of the form (2.2.2.1).

Example 2.2.4. Consider the quadric curve C (necessarily a conic) $x^2 - xy + y^2 = 3$. We will see that this is an ellipse, and we will work out is major and minor axes.

The quadratic form $Q(x, y) = x^2 - xy + y^2$ is of the form $Q_A(x, y)$ where $A = \left[\begin{smallmatrix} 1 & -1/2 \\ -1/2 & 1 \end{smallmatrix}\right]$. C is an ellipse if and only if A is positive definite, for in that case, after an orthogonal transformation, it is of the form $\lambda_1 (x')^2 + \lambda_2 (y')^2 = 3$. Now, the determinant of A is positive and $a_{11} = 1 > 0$, and hence by Proposition 1.2.1, A is positive definite. Thus C is an ellipse.

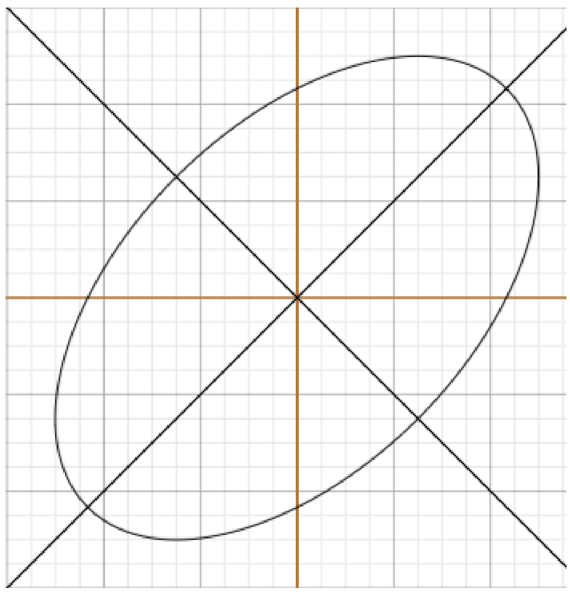
The characteristic polynomial of A is $\lambda^2 - 2\lambda + \frac{3}{4}$, and the eigenvalues are therefore $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$. An easy computation shows that the eigenspace of λ_1 is the set of points of the form (t, t), $t \in \mathbf{R}$, and the eigenspace corresponding to λ_2 is $(t, -t)$, $t \in \mathbf{R}$. It follows that $\boldsymbol{f}_1 = (\sqrt{2}^{-1}, \sqrt{2}^{-1})$ and $\boldsymbol{f}_2 = (-\sqrt{2}^{-1}, \sqrt{2}^{-1})$ gives us an orthonormal basis of $\mathbf{R}^2$ consisting of eigenvectors of A. Let $\Gamma = \left[\begin{smallmatrix} \sqrt{2}^{-1} & -\sqrt{2}^{-1} \\ \sqrt{2}^{-1} & \sqrt{2}^{-1} \end{smallmatrix}\right]$. Then setting

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \sqrt{2}^{-1} & -\sqrt{2}^{-1} \\ \sqrt{2}^{-1} & \sqrt{2}^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

we see that the equation of C in the $x'y'$ coordinates is $\frac{1}{2}(x')^2 + \frac{3}{2}(y')^2 = 3$, i.e.

$$\frac{(x')^2}{6} + \frac{(y')^2}{2} = 1.$$

As discussed earlier in item **1.** of §§2.2.2, such an equation is well known to be an ellipse with major axis of length $2\sqrt{6}$ along the x'-axis and the minor axis of length $2\sqrt{2}$ along the y' axis. The x'-axis is, in the old xy-system, the line $\{(t,t) \mid t \in \mathbf{R}\}$ and the y'-axis is the line $\{(t,-t) \mid t \in \mathbf{R}\}$. Here is a picture with the new axes also shown.



About these notes. This lecture was supposed to be given on March 18, 2020. Classes got suspended on March 17 because of the coronavirus COVID 19 pandemic, and all teaching moved online. These course notes are a reasonably faithful record of the lectures given (before the shutdown) at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.