## LECTURE 12

**Date of Lecture:** March 4, 2020

As always, $\mathbf{K} \in \{\mathbf{R},\, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An $n$-tuple $(x_1, \dots, x_n)$ of symbols ($x_i$ not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set $S$ to a product set $T_1 \times \cdots \times T_n$ will often be written as an $n$-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with $f_i$ a map from $S$ to $T_i$, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\|\ \|_2$ and we will simply denote it as $\|\ \|$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

### 1. Examples and counter examples (continued)

1.1. **Counterexamples (continued).** Here is the last of the counter examples that we were discussing last lecture.

**3.** <u>All directional derivatives of $f$ exist but $f$ is not differentiable.</u> Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^6 + y^2} & (x,y) \neq \mathbf{0} \\ \\ 0 & (x,y) = \mathbf{0}. \end{cases}$$

Let $\boldsymbol{u}$ be a unit vector. Then $\boldsymbol{u} = (\cos\theta, \sin\theta)$ for a unique $\theta \in [0,\, 2\pi)$. Clearly $D_{\boldsymbol{u}} f(\mathbf{0}) = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta)}{t}$, provided the limit on the right exists. If $\theta$ equals $0$ or $\pi$, then $f(t\cos\theta, t\sin\theta) = f(t\cos\theta, 0) = 0$ for all $t \in \mathbf{R}$, and hence $D_{\boldsymbol{u}} f(\mathbf{0}) = 0$ in these two cases. So now suppose $\theta \notin \{0, \pi\}$. Then $\sin\theta \neq 0$ and we have:

$$D_{\boldsymbol{u}} f(\mathbf{0}) = \lim_{t\to 0} \frac{1}{t} \frac{t^3 \cos^2\theta \sin\theta}{t^6\cos^6\theta + t^2 \sin^2\theta} = \lim_{t\to 0} \frac{t^3\cos^2\theta\sin\theta}{t^3(t^4\cos^6\theta + \sin^2\theta)} = \frac{\cos^2\theta}{\sin\theta}.$$

Thus $D_{\boldsymbol{u}}f(\mathbf{0})$ exists for all unit vectors $\boldsymbol{u}$. On $\mathbf{R}^2 \smallsetminus \{\mathbf{0}\}$, clearly $f$ is $\mathscr{C}^\infty$ and the directional derivative $D_{\boldsymbol{u}}f(\boldsymbol{p})$ exists for all $\boldsymbol{p} \in \mathbf{R}^2 \smallsetminus \{\mathbf{0}\}$ and all unit vectors $\boldsymbol{u}$.

Now, for $x \neq 0$,

$$f(x, x^3) = \frac{1}{2x}$$

and clearly $\lim_{x\to 0} f(x, x^3)$ does not exist. It follows that $f$ is not continuous at $\mathbf{0}$, and hence not differentiable there.

Another way of seeing that $f$ is not differentiable is to note that since $D_1 f(\mathbf{0}) = D_2 f(\mathbf{0})$ therefore if $Df(\mathbf{0})$ exists it must be $\mathbf{0}$ (since $(Jf)(\mathbf{0}) = 0$). In particular $\boldsymbol{\nabla} f(\mathbf{0}) = \mathbf{0}$. Then $D_{\boldsymbol{u}}f = \langle \boldsymbol{\nabla} f(\mathbf{0}), \boldsymbol{u}\rangle = \langle \mathbf{0}, \boldsymbol{u}\rangle = 0$. However, as we have seen $D_{\boldsymbol{u}}f(\mathbf{0}) \neq 0$ for many unit vectors $\boldsymbol{u}$. For example, if $\boldsymbol{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, then $D_{\boldsymbol{u}}f(\mathbf{0}) = \frac{1}{\sqrt{2}}$.

## 2. **Inner Product spaces**

### 2.1. **The orthogonal complement.**

By the *usual inner product*, or the *standard inner product* on $\mathbf{K}^n$ we mean the inner product given by

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$$

where $\bar{y}_i$ is the complex conjugate of $y_i$. The default inner product on $\mathbf{K}^n$, one we may often not specify, is this one.

Here are a couple of important definitions.

**Definition 2.1.1.** A mapping $f\colon X \to Z$ between $\mathbf{K}$ vector spaces is said to be *anti linear* if $f(ax + by) = \bar{a}f(x) + \bar{b}f(y)$ for $a, b \in K$ and $x, y \in X$. Note that if $\mathbf{K} = \mathbf{R}$,then anti linear maps are the same as linear maps. An *anti linear* isomorphism is an anti linear map which is an isomorphism on the underlying abelian groups.

**Definition 2.1.2.** Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional inner product space over $\mathbf{K}$, where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Let $H$ be a subspace of $V$. The *orthogonal complement* $H^\perp$ of $H$ is the subset

$$H^\perp = \{v \in V \mid \langle v, h\rangle = 0 \text{ for all } h \in H\}.$$

**Proposition 2.1.3.** *Let $V$ be a finite dimensional inner product space over $\mathbf{K}$ and $H$ a subspace of $V$. Then,*

(a) *$H^\perp$ is a subspace of $V$.*
(b) *$H^\perp = 0$ if and only if $H = V$.*
(c) *$H^{\perp\perp} = H$.*

*Proof.* This is Problem 1 of Homework 5. □

**Definition 2.1.4.** Let $V$ be a finite dimensional inner product space over $\mathbf{K}$. A basis $e_1, \dots, e_n$ of $V$ is called an *orthogonal basis* if $\langle e_i, e_j\rangle = 0$ for $i \neq j$, $i, j \in \{1, \dots, n\}$. It is called an orthonormal basis if $\langle e_i, e_j\rangle = \delta_{ij}$ for $i, j \in \{1, \dots, n\}$.

Note that an orthonormal basis is necessarily an orthogonal basis, but a orthogonal basis $e_1, \dots, e_n$ need not be an orthonormal basis. However, in this case, $\frac{1}{\|e_1\|}e_1, \dots, \frac{1}{\|e_n\|}e_n$ is an orthonormal basis.

**Lemma 2.1.5.** [Gram-Schmidt] *Let $n \in \mathbf{N}$ and $V$ be an inner product space over $\mathbf{K}$ with ordered basis $v_1, \dots, v_n$. Let $V_0 = \{0\}$ and for $k \in \{1, \dots, n\}$, let $V_k$ be the linear span of $\{v_1, \dots, v_k\}$. Then there exist $e_1, \dots, e_n \in V$ such that*

(a) $e_k \in V_k \cap V_{k-1}^{\perp}$, $k = 1, \dots, n$;
(b) $\|e_k\| = 1$ *for* $k = 1, \dots, n$.

*Moreover any such sequence of vectors $e_1, \dots, e_n$ is a basis of $V$.*

*Proof.* The last assertion is easily disposed. In fact one can prove a (seemingly) stronger statement, namely, if $1 \leq k \leq n$ and we have found $e_1, \dots, e_k$ such that $e_j \in V_j \cap V_{j-1}^{\perp}$, $j = 1, \dots, k$, and $\|e_j\| = 1$ for $1 \leq j \leq k$, then $e_1, \dots, e_k$ is a basis for $V_k$. Indeed, suppose $i$ and $j$ are distinct indices in $\{1, \dots, k\}$. Without loss of generality, we assume $i < j$. Then clearly $e_i \in V_{j-1}$ and hence $\langle e_i, e_j \rangle = 0$. Since $\|e_i\| = 1$ for all $i$, we see that $\langle e_i, e_j \rangle = \delta_{ij}$. If $c_1, \dots, c_k \in K$ are such that $\sum_{i=1}^{k} c_i e_i = 0$, then we have

$$c_j = \sum_{i=1}^{k} c_i \delta_{ij} = \sum_{i=1}^{k} c_i \langle e_i, e_j \rangle = \langle \sum_{i=1}^{k} c_i e_i, e_j \rangle = 0 \qquad (j = 1, \dots, k).$$

Thus $e_1, \dots, e_k$ are linearly independent vectors in the $k$-dimensional space $V_k$. Hence they must form a basis.

Set $e_1 = v_1/\|v_1\|$. Suppose $1 \leq k < n$ and we have found $e_1, \dots, e_k$ such that $e_j \in V_j \cap V_{j-1}^{\perp}$, $j = 1, \dots, k$, and $\|e_j\| = 1$ for $1 \leq j \leq k$. Let $w_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle e_i$. Then

$$\begin{aligned}\langle w_{k+1}, e_j \rangle = \langle v_{k+1}, e_j \rangle - \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle \langle e_i, e_j \rangle &= \langle v_{k+1}, e_j \rangle - \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle \delta_{ij} \\ &= \langle v_{k+1}, e_j \rangle - \langle v_{k+1}, e_j \rangle \\ &= 0\end{aligned}$$

Thus $w_{k+1}$ is orthogonal to every element in a basis of $V_k$. It follows that $w_{k+1} \in V_k^{\perp}$. Now $w_{k+1} \notin V_k$, for if $w_{k+1}$ lies in $V_k$ then $v_{k+1} = w_{k+1} + \sum_{i=1}^{k} \langle v_{k+1}, e_i \rangle e_i$ must also lie in $V_k$, which is not possible, for $v_1, \dots, v_n$ is a basis for $V$. In particular $w_{k+1} \neq 0$. Set $e_{k+1} = w_{k+1}/\|w_{k+1}\|$. Then $e_{k+1} \in V_k^{\perp} \cap V_{k+1}$ and $\|e_{k+1}\| = 1$. By induction we have found $e_i, \dots, e_n$ meeting the specifications of the theorem. □

**Theorem 2.1.6.** [Gram-Schmidt] *Every finite dimensional inner product space over $\mathbf{K}$ has an orthonormal basis.*

*Proof.* This is immediate from Lemma 2.1.5 □

**Remark 2.1.7.** The process of extracting an orthonormal basis from a given basis described in Lemma 2.1.5 is called the *Gram-Schmidt orthogonalisation process* (or, in US spelling, *Gram-Schmidt orthogonalization process*).

**Proposition 2.1.8.** *Let $H$ be a subspace of a finite dimensional inner product space $V$ over $\mathbf{K}$. Then $V = H \oplus H^{\perp}$, i.e. every $v \in V$ has a unique decomposition $v = v' + v''$ such that $v' \in H$ and $v'' \in H^{\perp}$.*

*Proof.* This is Problem 2 of Homework 5. □

**Proposition 2.1.9.** *Let $d$ and $m$ be non-negative integers and $n = d+m$. Consider $\mathbf{K}^n$ with the standard inner product and identify $\mathbf{K}^n$ with $\mathbf{K}^d \times \mathbf{K}^m$ in the usual way. Let $\pi\colon \mathbf{K}^n \to \mathbf{K}^d$ and $\mathbf{K}^n \to \mathbf{K}^m$ be the projection maps given by $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$ and $\pi'(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$. Let $H$ be a $d$-dimensional subspace of $\mathbf{K}^n$. Then $\pi(H) = \mathbf{R}^d$ if and only if $\pi'(H^\perp) = \mathbf{R}^m$.*

*Proof.* By symmetry (since $H^{\perp\perp} = H$) it is enough to prove that if $\pi(H) = \mathbf{K}^d$ then $\pi'(H^\perp) = \mathbf{K}^m$. According to Proposition 2.1.8, $\mathbf{K}^n = H \oplus H^\perp$, and hence $\dim_{\mathbf{K}} H^\perp = m$. Thus $\pi'(H^\perp) = \mathbf{K}^m$ if and only if $\pi'|_{H^\perp}\colon H^\perp \to \mathbf{K}^m$ is an isomorphism.

Suppose $\pi(H) = \mathbf{K}^d$ and suppose $\boldsymbol{v} \in H^\perp$ is such that $\pi'(\boldsymbol{v}) = \{\mathbf{0}\}$. Then $\boldsymbol{v} = (\boldsymbol{x}, \mathbf{0})$. Since $\pi(H) = \mathbf{K}^d$,there exists $\boldsymbol{h} \in H$ be such that $\pi(\boldsymbol{h}) = \boldsymbol{x}$. In fact, since $\dim_{\mathbf{K}} H = d$, $\pi|_H$ is an isomorphism, and the element $\boldsymbol{h}$ is the unique element in $H$ such that $\pi(\boldsymbol{h}) = \boldsymbol{x}$. Clearly $\boldsymbol{h} = (\boldsymbol{x}, \boldsymbol{y})$ for some $\boldsymbol{y} \in \mathbf{K}^m$. Since $\boldsymbol{v} \in H^\perp$, we have $\langle \boldsymbol{v}, \boldsymbol{h} \rangle = 0$. Thus $\|\boldsymbol{x}\|^2 = \langle (\boldsymbol{x}, \mathbf{0}), (\boldsymbol{x}, \boldsymbol{y}) \rangle = \langle \boldsymbol{v}, \boldsymbol{h} \rangle = 0$. This means $\boldsymbol{x} = \mathbf{0}$, i.e. $\boldsymbol{v} = \mathbf{0}$. It follows that $\pi'|_{H^\perp}\colon H^\perp \to \mathbf{K}^m$ is injective. Matching the dimensions of $H^\perp$ and $\mathbf{K}^m$, we see that $\pi'|_{H^\perp}$ is an isomorphism. □

## 3. Matrix representations

3.1. **Complexification.** This rightly ought to be addressed through tensor products. However, since you probably do not know what a tensor product of modules over a ring (or of vector spaces over a field) is, here is a poor person's version of complexification. We restrict ourselves to linear endomorphisms of $\mathbf{R}^n$.

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation, and $A$ the $n \times n$ matrix of $T$ with respect to the standard basis on $\mathbf{R}^n$. The *complexification* $T_{\mathbf{C}}$ of $T$ is the $\mathbf{C}$-linear map

$$T_{\mathbf{C}}\colon \mathbf{C}^n \longrightarrow \mathbf{C}^n \tag{3.1.1}$$

given by the formula

$$T_{\mathbf{C}}(\boldsymbol{x}) = A\boldsymbol{x}, \qquad (\boldsymbol{x} \in \mathbf{C}^n) \tag{3.1.2}$$

where the right side of the equality is matrix multiplication. It is obvious that the matrix of $T_{\mathbf{C}}$ with respect the standard basis of $\mathbf{C}^n$ is also $A$.

**About these notes.** These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.