LECTURE 11

Date of Lecture: March 2, 2020

As always, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

An n-tuple $(x_1, \dots, x_n)$ of symbols (x_i not necessarily real or complex numbers) will also be written as a column vector when convenient. Thus

$$(x_1, \dots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

A map $\boldsymbol{f}$ from a set S to a product set $T_1 \times \cdots \times T_n$ will often be written as an n-tuple $\boldsymbol{f} = (f_1, \dots, f_n)$, with f_i a map from S to T_i, and hence, by the above convention, as a column vector

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(See Remark 2.2.2 of Lecture 5.)

The default norm on Euclidean spaces of the form $\mathbf{R}^n$ is the Euclidean norm $\| \, \|_2$ and we will simply denote it as $\| \, \|$.

☡ Note that $(x_1, \dots, x_n) \neq [x_1 \dots x_n]$. Each side is the transpose of the other.

1. Directional derivatives

Let $\boldsymbol{u} \in \mathbf{R}^n$ be a unit vector, i.e. $\|\boldsymbol{u}\| = 1$. An obvious generalisation of the notion of a partial derivative with respect to a variable is the the so-called *directional derivative along* $\boldsymbol{u}$ of a function f at a point $\boldsymbol{p}$

1.1. . Let $\boldsymbol{f}\colon U \to \mathbf{R}^m$ be a map with U an open subset of $\mathbf{R}^n$. Let $\boldsymbol{p} \in U$ and let $\boldsymbol{u} \in \mathbf{R}^n$ be a unit vector. The *directional derivative* $D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p}) \in \mathbf{R}^m$ *of* $\boldsymbol{f}$ *along* $\boldsymbol{u}$ *at* $\boldsymbol{p}$ is

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p}) = \lim_{h \to 0} \frac{\boldsymbol{f}(\boldsymbol{p} + h\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{p})}{h} \tag{1.1.1}$$

provided the limit exists. Otherwise we say that the directional derivative of $\boldsymbol{f}$ along $\boldsymbol{u}$ does not exist at $\boldsymbol{p}$, or simply that $D_{\boldsymbol{u}}\boldsymbol{f}$ does not exist at $\boldsymbol{p}$.

Lemma 1.1.2. *Let U be an open set in $\mathbf{R}^n$, $\boldsymbol{p}$ a point in U and $\boldsymbol{f}\colon U \to \mathbf{R}^m$ a map which is differentiable at $\boldsymbol{p}$. Then for for every unit vector $\boldsymbol{u} \in \mathbf{R}^n$, the directional derivative $D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p})$ exists and*

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{f}'(\boldsymbol{p})\boldsymbol{u}.$$

Proof. Let $\delta > 0$ be small enough that $B(\boldsymbol{p}, \delta) \subset U$. Define $\boldsymbol{\gamma}\colon (-\delta, \delta) \to U$ by the formula $\boldsymbol{\gamma}(t) = \boldsymbol{p} + t\boldsymbol{u}$. Now

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p}) = \lim_{h\to 0} \frac{\boldsymbol{f}(\gamma(h)) - \boldsymbol{f}(\gamma(0))}{h} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{f}\circ\boldsymbol{\gamma})(t)\bigg|_{t=0}.$$

The chain rule then gives $D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{p}) = \boldsymbol{f}'(\boldsymbol{p})\boldsymbol{\gamma}'(0) = \boldsymbol{f}'(\boldsymbol{p})\boldsymbol{u}$. □

Remark 1.1.3. Suppose U and $\boldsymbol{p}$ are as above. Let $f\colon U \to \mathbf{R}$ be a map which is differentiable at $\boldsymbol{p}$. Then $f'(\boldsymbol{p})$ is a $1 \times n$ matrix, and $\boldsymbol{\nabla} f(\boldsymbol{p}) \in \mathbf{R}^n$. The Lemma can be re-written in this case as

(1.1.3.1) $$D_{\boldsymbol{u}}f(\boldsymbol{p}) = \langle \boldsymbol{\nabla} f(\boldsymbol{p}), \boldsymbol{u}\rangle \qquad (\boldsymbol{u}\in\mathbf{R}^n, \|\boldsymbol{u}\| = 1).$$

The Cauchy-Schwarz inequality then gives

(1.1.3.2) $$|D_{\boldsymbol{u}}f(\boldsymbol{p})| \le \|\boldsymbol{\nabla} f(\boldsymbol{p})\| \qquad (\boldsymbol{u}\in\mathbf{R}^n, \|\boldsymbol{u}\| = 1).$$

Finally if $f'(\boldsymbol{p}) \neq 0$, i.e., $\boldsymbol{\nabla} f(\boldsymbol{p}) \neq \mathbf{0}$, then setting $\boldsymbol{u} = \boldsymbol{\nabla} f(\boldsymbol{p})/\|\boldsymbol{\nabla} f(\boldsymbol{p})\|$ in (1.1.3.1) we see that

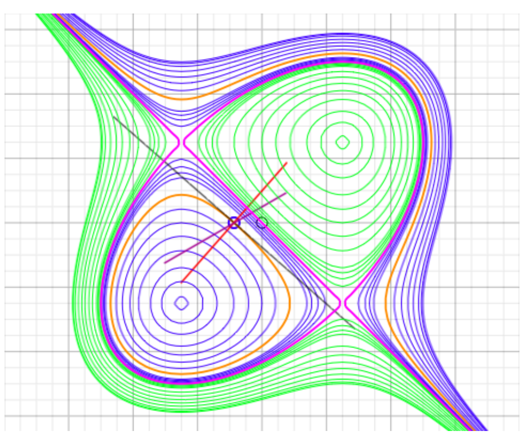
(1.1.3.3) $$D_{\boldsymbol{u}}f(\boldsymbol{p}) = \|\boldsymbol{\nabla} f(\boldsymbol{p})\| \quad \text{when } \boldsymbol{\nabla} f(\boldsymbol{p}) \neq \mathbf{0} \text{ and } \boldsymbol{u} = \boldsymbol{\nabla} f(\boldsymbol{p})/\|\boldsymbol{\nabla} f(\boldsymbol{p})\|.$$

The relations (1.1.3.1), (1.1.3.2) and (1.1.3.3) taken together are often informally phrased as "the gradient is the direction of the maximum rate of change of f".

Example 1.1.4. Let $f(x, y) = x^3+y^3-3x-3y$. Then $\boldsymbol{\nabla} f(x, y) = (3x^2-3, 3y^2-3)$. Now every unit vector in $\mathbf{R}^2$ is of the form $\boldsymbol{u}_\theta = (\cos\theta, \sin\theta)$ for a unique $\theta \in [0,\, 2\pi)$. From (1.1.3.1) we then see that

$$D_{\boldsymbol{u}_\theta} f(x, y) = (3x^2 - 3)\cos\theta + (3y^2 - 3)\sin\theta.$$

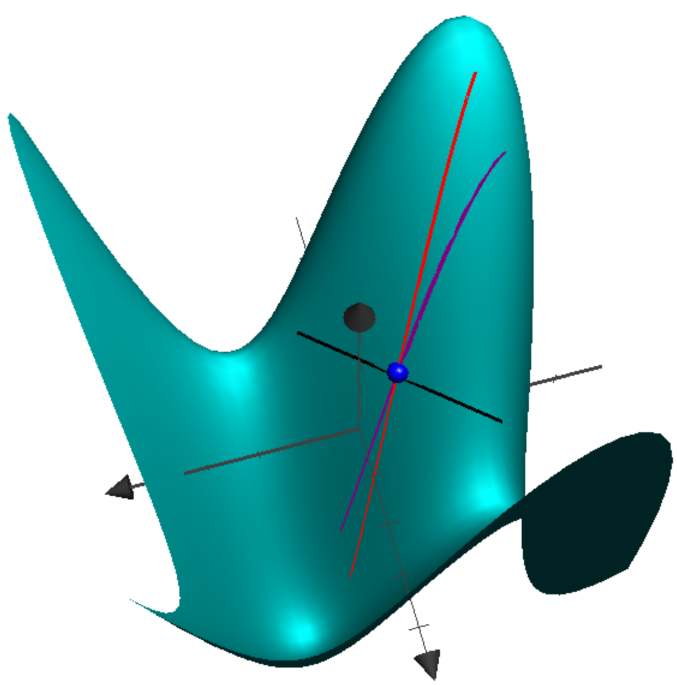
In the picture below the non straight lines are curves with equations of the form $f(x, y) = c$, i.e. $x^3 + y^3 - 3x - 3y = c$. These are the so-called *level curves* of f. The orange curves correspond to $f(x, y) = 1$. The blue curves are $f(x, y) = c$ for $c = 0.125$, 0.25, 0.375, 0.5, 1.5, 2, 2.5, 3, 3.5. 3.8, and 3.98. The green ones are for negative values of c, specifically $c = -0.125$, -0.25, -0.375, -0.5, -1, -1.5, -2, -2.5, and -3. The outer most closed loop in blue is for $c = 0.125$ and the innermost closed blue loop is for $c = 3.98$ whereas for the green loops the outermost corresponds to $c = -0.125$ and the innermost to $c = -3.98$.



The magenta curve is the curve $f(x, y) = 0$. It is clear that $f(x, y) = (x + y)(x^2 - xy + y^2 - 3)$ and hence the curve $f(x, y) = 0$ is the union of the line $x + y = 0$ and the ellipse $x^2 - xy + y^2 = 3$. This is seen in the picture too.

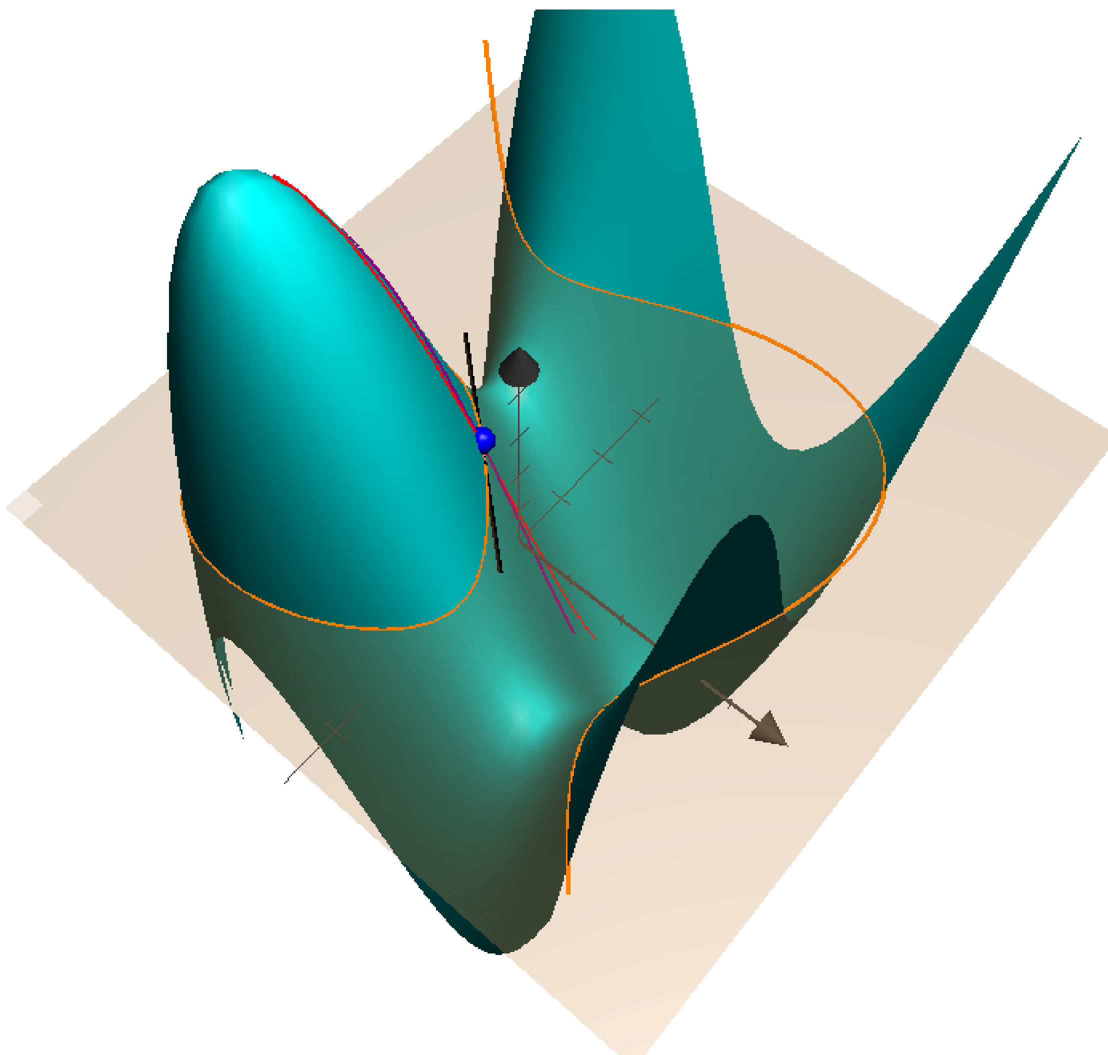
On the curve $f(x, y) = 1$ we have a point $\boldsymbol{p}$ whose y co-ordinate is 0 and whose x co-ordinate is (up to four decimal places) is -0.3473. The function f is constant on each level curve. The direction of the normal vector at $\boldsymbol{p}$ to $f = 1$ is given by the red line. This, from our theory, is the direction of $\boldsymbol{\nabla} f(\boldsymbol{p})$. The purple line corresponds to the direction $\boldsymbol{u}_\theta$, where $\theta = \pi/6$. A look at the picture shows that the rate of change of f is indeed the greatest as one travels along the red line. The rate of change of f along the purple line is lower than that along the red line as a visual inspection shows. The black straight line through $\boldsymbol{p}$ is the tangent line to the level curve $f(x, y) = 1$ at $\boldsymbol{p}$. That is the direction of the slowest rate of change of f. And this makes sense because f is a constant on the level curve, so it indeed has a very slow rate of change along it (to put it mildly).

Below is the graph of the surface $z = x^3 + y^3 - 3x - 3y$. To orient yourself to the planar the"contour maps" above, remember that in the contour maps the closed loops depict higher values of f as you move "inwards" in the southeast direction, and so the hill in the 3D graph below corresponds to that. Another way you can orient yourself is to note that in the 3D graph, the left (and slightly downward) pointing arrow is the positive x-axis, and the upward arrow the z-axis. If you imagine a varying plane $z = c$ (this is the equation of a plane parallel to the xy-plane), then as c varies, this plane cuts the surface in the various level curves (the "contours").



Use your visual skills to make the correspondence. It will be worth your while to spend (more than) a few minutes shoring up your visual intuition. The blue dot is the lift to the surface of the point of interest in the planar graphs given above (the "contour map"). The red curve is the lift to the surface of the red line of the planar contour map, the purple curve the lift of the purple line, and finally the black line corresponds to a tangential direction at the point of interest to a level curve, just as it did in the planar contour maps. You can see that the red curve is the path of steepest ascent and descent at the point of interest. In fact the gradient (the term is being used as it is in everyday language) is so steep that it looks like almost like a straight line. And it also shows you why the mathematical entity $\boldsymbol{\nabla} f(\boldsymbol{p})$ is called the gradient. It corresponds to the term used in everyday language.

Below is another view of the surface together with the plane $z = 1$. The intersection of the plane $z = 1$ with the graph of $z = f(x, y)$ gives us a curve (the orange one) which is the replica of the curve $x^3 + y^3 - 3x - 3y = 1$ on the plane (namely a replica of the orange curve on the plane graph) but at a "height" of $z = 1$:



2. **Mixed partial derivatives**

2.1. **The two variables case.** Throughout this subsection we assume

$$f\colon U \to \mathbf{R}$$

is a map with U an open set in $\mathbf{R}^2$, with $D_1 f$, $D_2 f$, $D_{21} f$ existing on U and (a, b) a point in U. For h, k real numbers $Q(h, k)$ will denote the closed rectangle which has(a, b) and $(a + h, b + k)$ as diagonally opposite vertices. If $Q = Q(h, k)$ lies entirely in U set

$$\Delta(f, Q) := f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b). \tag{2.1.1}$$

We then have the following generalisation of the mean value theorem (see [R, Chapter 9, pp. 235–236]).

Lemma 2.1.2. *Let $Q = Q(h, k)$ lie in U. Them there exists a point (s, t) in the interior of Q such that*

$$\Delta(f, Q) = hkD_{21} f(s, t).$$

Proof. Let $u(x) = f(x, b + k) - f(x, b)$. Then

$$\Delta(f, Q) = u(a + h) - u(a).$$

By the mean value theorem for one variable, there exists s in the open interval with end points a and $a + h$ such that

$$\Delta(f, Q) = hu'(s) = h(D_1 f(s, b + k) - D_1 f(s, b)).$$

Applying the mean value for one variable again, this time on the difference $D_1 f(s, b+ k) - D_1 f(s, b)$ we see that there exists t in the open interval whose end points are b and $b + k$ such that $D_1 f(s, b + k) - D_1 f(s, b) = kD_{21} f(s, t)$. This gives

$$\Delta(f, Q) = hkD_{21} f(s, t)$$

as required. □

Theorem 2.1.3. *Suppose $D_{21} f$ is continuous at (a, b). Then $D_{12} f(a, b)$ exists and*

$$D_{12} f(a, b) = D_{21} f(a, b).$$

Proof. Let $\epsilon > 0$ be given. The continuity of $D_{21} f$ at (a, b) implies there exists $\delta > 0$ such that if h, k are non-zero with $\max(|h|, |k|) < \delta$,

$$|D_{21} f(x, y) - D_{21} f(a, b)| < \epsilon \qquad ((x, y) \in Q(h, k)).$$

Let (h, k) be as above and let $Q = Q(h, k)$. By Lemma 2.1.2 this means that for

$$\left| \frac{\Delta(f, Q)}{hk} - D_{21} f(a, b) \right| < \epsilon.$$

Letting $k \to 0$, the above gives

$$\left| \frac{D_2 f(a + h, b) - D_2 f(a, b)}{h} - D_{21} f(a, b) \right| \le \epsilon \qquad (0 < |h| < \delta)$$

It follows that $\lim_{h\to 0} \frac{1}{h}(D_2 f(a+h, b) - D_2 f(a, b))$ exists and equals $D_{21} f(a, b)$. In other words $D_{12} f(a, b)$ exists and equals $D_{21} f(a, b)$. □

Corollary 2.1.4. *If f is in $\mathscr{C}^2$ then $D_{12} f = D_{21} f$.*

Proof. This is immediate from the theorem. □

2.2. **Several variables and higher order mixed partials.** In most applications, it is the corollary above that is important.

Theorem 2.2.1. *Fix $k, n, m \in \mathbf{N}$. Let U be an open subset of $\mathbf{R}^n$ and $\boldsymbol{f}\colon U \to \mathbf{R}^m$ a $\mathscr{C}^k$ map. Let $(i_1, i_2, \dots, i_k) \in \{1, \dots, n\}^k$. Then*

$$D_{j_1 \dots j_k} \boldsymbol{f} = D_{i_1 \dots i_k} \boldsymbol{f}$$

for every permutation $(j_1, j_2, \dots, j_k)$ of $(i_1, i_2, \dots, i_k)$. (We say $(j_1, j_2, \dots, j_k)$ is a permutation of $(i_1, i_2, \dots, i_k)$ if there exists an element σ in S_k, the symmetric group on $\{1, \dots, k\}$, such that $j_s = i_{\sigma(s)}$.)

Proof. For $k = 1$ there is nothing to prove. For $k = 2$, by keeping $n - 2$ variables fixed, it is clear from Corollary 2.1.4 that if $i, j \in \{1, \dots, n\}$ then $D_{ij}\boldsymbol{f} = D_{ji}\boldsymbol{f}$ (if $i = j$, clearly $D_{ij}\boldsymbol{f} = D_{ji}\boldsymbol{f}$). Since $(j_1, \dots, j_k)$ can be obtained from $(i_1, \dots, i_k)$ by successive transpositions of the form $(l, l + 1)$, we are done. □

3. Examples and counter examples

To recognise a function as a $\mathscr{C}^k$ function, it is worth remembering that the sum, product, difference, composites and ratios (where the denominator does not vanish) of $\mathscr{C}^k$ functions are $\mathscr{C}^k$. This follows easily from looking at partial derivatives, which is essentially one variable calculus, and the chain rule for composite functions.

In particular polynomial functions in several variables are $\mathscr{C}^\infty$ and hence by Theorem 2.2.1 the order of differentiating does not matter for polynomials when taking mixed partial derivatives.

3.1. Examples.

1. Let $f(x, y) = x^3 y \sin(xy) - e^x \cos(x + y)$. Then f is a difference of two functions each of which is a product of functions which are either $\mathscr{C}^\infty$ or composites of C^∞ functions. For example, $x^3 y \sin(xy)$ is the product of x, x, x, y, and $\sin(xy)$, and $\sin(xy)$ is the composite of sin and xy. Therefore $x^3 y \sin(xy)$ is C^∞. Similarly $e^x \cos(x + y)$ is $\mathscr{C}^\infty$. Thus f is $\mathscr{C}^\infty$ on $\mathbf{R}^2$. In particular f is $\mathscr{C}^2$ Therefore according to Theorem 2.2.1, or Corollary 2.1.4, we must have $\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial^2 f(x,y)}{\partial x \partial y}$. One can verify this by direct computations. Indeed

$$\begin{aligned}
\frac{\partial^2 f(x, y)}{\partial y \partial x} &= \frac{\partial}{\partial y}\Big\{3x^2 y \sin(xy) + x^3 y^2 \cos(xy) - e^x \cos(x + y) + e^x \sin(x + y)\Big\} \\
&= 3x^2 \sin(xy) + 3x^3 y \cos(xy) + 2x^3 y \cos(xy) - x^4 y^2 \sin(xy) \\
&\qquad + e^x \sin(x + y) + e^x \cos(x + y) \\
&= (3x^2 - x^4 y^2) \sin(xy) + 5x^3 y \cos(xy) \\
&\qquad + e^x \sin(x + y) + e^x \cos(x + y).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{\partial^2 f(x, y)}{\partial x \partial y} &= \frac{\partial}{\partial x}\Big\{x^3 \sin(xy) + x^4 y \cos(xy) + e^x \sin(x + y)\Big\} \\
&= 3x^2 \sin(xy) + x^3 y \cos(xy) + 4x^3 y \cos(xy) - x^4 y^2 \sin(xy) \\
&\qquad + e^x \sin(x + y) + e^x \cos(x + y) \\
&= (3x^2 - x^4 y^2) \sin(xy) + 5x^3 y \cos(xy) \\
&\qquad + e^x \sin(x + y) + e^x \cos(x + y).
\end{aligned}$$

It is clear that $\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial^2 f(x,y)}{\partial x \partial y}$, as predicted by Corollary 2.1.4 (or Theorem 2.2.1).

2. Suppose P and Q are $\mathscr{C}^1$ functions on an open set U in $\mathbf{R}^2$. One can ask if there is a function $f\colon U \to \mathbf{R}$ such that $D_1 f = P$ and $D_2 f = Q$. If such an f exists, then $D_{11} f = D_1 P$, $D_{21} f = D_2 P$, $D_{12} f = D_1 Q$, and $D_{22} f = D_2 Q$ are all continuous. Hence f is $\mathscr{C}^2$. According to Corollary 2.1.4 $D_{21} f = D_{12} f$, i.e. $D_2 P = D_1 Q$.

For example, if $P(x,y) = e^{xy} \sin x$ and $Q(x,y) = e^y \cos x$ then $D_2 P(x,y) = xe^{xy} \sin x$ and $D_1 Q(x,y) = -e^y \sin x$. Clearly $D_2 P \neq D_1 Q$. It follows that there is no f such that $D_1 f = P$ and $D_2 f = Q$.

What is interesting is that if $D_2 P = D_1 Q$ and U is convex (or more generally *simply connected*) then in fact there is a function f, necessarily $\mathscr{C}^2$, such that $D_1 f = P$ and $D_2 f = Q$. This rests on the so-called *Green's Theorem* connecting path integrals with plane integrals which you will learn about in your Calculus course next semester. These ideas also occur in the differential equations course in your fourth semester.

Exercise: Let $P(x,y) = e^x \cos y + 3x^2 y^2$ and $Q(x,y) = -e^x \sin y + 2x^3 y$. Then $\frac{\partial P(x,y)}{\partial y} = -e^x \sin y + 6x^2 y = \frac{\partial Q(x,y)}{\partial x}$. Assuming the result in the last paragraph, there exists a $\mathscr{C}^2$ function f on $\mathbf{R}^2$ such that $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. Find such an f.

3.2. **Counter examples.** For ease of understanding, we consider real valued functions of two variables, and the domain will be clear from the formulas defining the functions. In all the counter examples below f is continuous, except in Example **3.**

1. $D_1 f$ and $D_2 f$ exist but f is not differentiable. We know from Theorem 2.2.1 of Lecture 8 that if $D_1 f$ and $D_2 f$ exist and are continuous then f is differentiable. So any example we construct must have at least one partial derivative which is discontinuous. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ \\ 0, & (x,y) = (0,0). \end{cases}$$

Let us verify that f is continuous. The only difficulty is at the origin, for on $\mathbf{R}^2 \smallsetminus \{(0,0)\}$ it is the ratio of continuous functions where the denominator does not vanish. Now, $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$. Hence, for $(x,y) \neq (0,0)$,

$$|f(x,y) - f(0,0)| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}.$$

It follows that $|f(x,y) - f(0,0)| \to 0$ as $(x,y) \to (0,0)$. Thus f is continuous.

If $(x,y) \neq (0,0)$, it is clear that $D_1 f(x,y)$ and $D_2 f(x,y)$ exist. Indeed, for such (x,y)

$$D_1 f(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}}, \quad \text{and} \quad D_2 f(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

Now

$$\frac{f(h,0) - f(0,0)}{h} = 0$$

for all $h \neq 0$, whence, letting $h \to 0$, we see that $D_1 f(0,0)$ exists. In fact $D_1 f(0,0) = 0$. By symmetry $D_2 f(0,0)$ exists and is zero. Thus the partial derivatives of f exist for all points in $\mathbf{R}^2$.

We claim that f is not differentiable at $(0,0)$. Suppose f is differentiable at $\mathbf{0}$. Since $D_1 f(\mathbf{0}) = D_2 f(\mathbf{0}) = 0$, we have $Jf(\mathbf{0}) = 0$, whence $f'(\mathbf{0}) = 0$. This means for $\boldsymbol{h} \neq \mathbf{0}$,

$$\lim_{\boldsymbol{h} \to \mathbf{0}} \frac{f(\mathbf{0} + \boldsymbol{h}) - f(\mathbf{0})}{\|\boldsymbol{h}\|} = 0.$$

Now for $\boldsymbol{h} = (h, h)$, $h \neq 0$ we have

$$\frac{|f(\mathbf{0} + \boldsymbol{h}) - f(\mathbf{0})|}{\|\boldsymbol{h}\|} = \frac{|f(h,h)|}{\|\boldsymbol{h}\|} = \frac{1}{2}$$

and hence $\lim_{\boldsymbol{h} \to \mathbf{0}} \frac{1}{\|\boldsymbol{h}\|}(f(\mathbf{0} + \boldsymbol{h})) - f(\mathbf{0}))$ cannot be zero.

In particular, at least one of the partial derivatives of f is not continuous at $\mathbf{0}$ (as a matter of fact, since f is symmetric in x and y, both partial derivatives are discontinuous at $\mathbf{0}$). We can verify this directly, for when $y \neq 0$

$$D_1 f(0, y) = \frac{y^3}{(0^2 + y^2)^{3/2}} = \frac{y^3}{|y|^3} = \operatorname{sign}(y),$$

showing that $D_1 f$ is not continuous on the y-axis.

2. <u>Df exists but is not continuous.</u> Recall a well known example from one variable Calculus, namely the function $g \colon \mathbf{R} \to \mathbf{R}$ given by

$$g(x) := \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The function g is a famous example of a function which is differentiable and whose derivative is not continuous. In fact

$$g'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

It is clear that g' is not continuous at 0 since the limit of $g'(x)$ as $x \to 0$ does not exist, but $g'(0) = 0$. Define f by the formula

$$f(x, y) = g(\sqrt{x^2 + y^2}).$$

It is clear f is continuous since it is the composite of two continuous maps. Similarly on $\mathbf{R}^2 \setminus \{\mathbf{0}\}$, f is differentiable by the chain rule. It is also differentiable at $\mathbf{0}$ since $\frac{1}{\|\boldsymbol{h}\|}|f(\boldsymbol{h}) - f(\mathbf{0})| \leq \|\boldsymbol{h}\|^2/\|\boldsymbol{h}\| = \|\boldsymbol{h}\| \to 0$ as $\boldsymbol{h} \to \mathbf{0}$, giving that $Df(0,0)$ exists and is zero. It follows that $(Jf)(0,0) = 0$, whence $D_1 f(0,0) = D_2 f(0,0) = 0$.

If Df is continuous, then $D_i f$ is continuous for $i = 1, 2$. However, for $x > 0$, $D_1 f(x, 0) = g'(x) = 2x \sin(1/x) - \cos(1/x)$ which does not have a limit as $x \to 0$. Thus $D_1 f$ is not continuous at $\mathbf{0}$. It follows that neither is Df.

3. <u>All directional derivatives of f exist but f is not differentiable.</u> Will do this next lecture.

About these notes. These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.

References

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