LECTURE 1

Date of Lecture: January 6, 2020

1. Normed linear spaces

Throughout $\mathbf{K}$ will denote one of $\mathbf{R}$ or $\mathbf{C}$. If X is a set, then a *sequence in* X is a map $S\colon \mathbf{N} \to X$. If S is a sequence in X and $x_n = S(n)$ for each $n \in \mathbf{N}$, then we commonly denote S by the symbol $\{x_n\}$, or sometimes by $\{x_n\}_{n=1}^{\infty}$, or by $\{x_n\}_{n\in\mathbf{N}}$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1.1. Norms.

Let V be a K-vector space. A *norm* $\|\cdot\|$ on V is a map

$$\|\cdot\|\colon V \to [0,\infty)$$

such that

1. $\|x\| = 0$ if and only if $x = 0$;
2. $|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbf{K}$ and every $x \in V$;
3. $\|x+y\| \leq \|x\| + \|y\|$ for every $x, y \in V$.

It is easy to see that if $\|\cdot\|$ is a norm on $\mathbf{K}$-vector space V, then

$$d\colon V \times V \to [0,\infty), \quad (v,w) \mapsto \|v-w\|$$

defines a metric on V.

Examples 1.1.2. Here are the most common examples of norms. Proofs that these are norms will be given in the next class, except for $\|\ \|_2$ in (1) below.

1. Let $\mathbf{K} = \mathbf{R}$ and $V = \mathbf{R}^n$. For each $p \in [1,\infty)$ define

   $$\|\boldsymbol{x}\|_p = \left\{\sum_{i=1}^{n} |x_i|_i^p\right\}^{1/p} \qquad (\boldsymbol{x} \in \mathbf{R}^n).$$

   Then $\|\cdot\|_p$ is a norm for each $p \in [1,\infty)$. For $p = \infty$ the definition is

   $$\|\boldsymbol{x}\|_\infty = \max_{1\leq i\leq n} |x_i| (\boldsymbol{x} \in \mathbf{R}^n).$$

   In all these examples, x_i is the i^{th}-co-ordinate of $\boldsymbol{x} \in \mathbf{R}^n$. We point out that

   $$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2$$

   for every $\boldsymbol{x} \in \mathbf{R}^n$. This is clear from the definitions for, $|x_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every i. The proof that $\|\ \|_2$ is a norm is given later in this lecture.
2. Let $\mathbf{K} = \mathbf{C}$, and use the formulas in (1) to define $\|\cdot\|_p$ for $1 \leq p \leq \infty$ on $\mathbf{C}^n$. Note that the inequality $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2$ continues to hold, and for the same reason as in (1).

(3) Let $I = [a, b]$ be a closed and bounded interval andlet $C(I)$ be the $\mathbf{R}$-vector space of continuous $\mathbf{R}$-valued functions on I. Since I is compact, every element of $C(I)$ is bounded (and achieves its bound). For $f \in C(I)$ define

$$\|f\|_\infty = \sup_{x\in I} |f(x)|.$$

Then $\|\cdot\|_\infty$ is a norm on $C(I)$. We point out that $C(I)$ is infinite dimensional.

1.1.3. Note that $\|\cdot\|_2$ is the usual Euclidean norm on $\mathbf{K}^n$. Let us prove it is a norm. Properties (1) and (2) are obvious and was done in class. Proving Property (3) is a little involved. We first prove the *Cauchy-Schwarz inequality*, namely if $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are points in $\mathbf{C}^n$ then

$$|x_1\bar{y_1} + \cdots + x_n\bar{y_n}| \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2. \tag{1.1.3.1}$$

This is proven as follows. First, for simplicity, use the symbol $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ for $\sum_{i=1}^n x_i\bar{y}_i$. Then (1.1.3.1) becomes

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2. \tag{1.1.3.2}$$

If $\boldsymbol{y} = \mathbf{0}$ the above inequality is obvious. So assume $\boldsymbol{y} \ne \mathbf{0}$. For $s \in \mathbf{R}$ consider

$$\begin{aligned} f(s) = \|\boldsymbol{x} - s\boldsymbol{y}\|_2^2 &= \sum_{i=1}^n |x_i - sy_i|^2 \\ &= \sum_{i=1}^n (x_i - sy_i)(\bar{x}_i - s\bar{y}_i) \\ &= \sum_{i=1}^n \Big\{ |x_i|^2 + s^2|y_i|^2 - 2s\mathrm{Re}(x_i\bar{y}_i) \Big\} \\ &= \|\boldsymbol{x}\|_2^2 + s^2\|\boldsymbol{y}\|_2^2 - 2s\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle. \end{aligned}$$

Then $f'(s) = 2s\|\boldsymbol{y}\|_2^2 - 2\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle$. The derivative vanishes at $s_0 = \mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle/\|\boldsymbol{y}\|_2^2$. This is clearly a point of minimum of f (note that the second derivative of f is positive at s_0, or alternately, note that the graph of f as a function of s is a parabola). Thus $f(s) \ge f(s_0)$ for all $s \in \mathbf{R}$. Since $f(s) = \|\boldsymbol{x} - s\boldsymbol{y}\|_2^2 \ge 0$ for all $s \in \mathbf{R}$, we have $f(s_0) \ge 0$. Thus

$$\begin{aligned} f(s_0) = \|\boldsymbol{x} - s_0\boldsymbol{y}\|_2^2 &= \|\boldsymbol{x}\|_2^2 + \left(\frac{\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle}{\|\boldsymbol{y}\|_2^2}\right)^2 \|\boldsymbol{y}\|_2^2 - 2\left(\frac{\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle}{\|\boldsymbol{y}\|_2^2}\right)\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\ &= \|\boldsymbol{x}\|_2^2 - \frac{(\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle)^2}{\|\boldsymbol{y}\|_2^2} \end{aligned}$$

Since $f(s_0) \ge 0$, the above equality gives

$$\|\boldsymbol{x}\|_2^2 - \frac{(\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle)^2}{\|\boldsymbol{y}\|_2^2} \ge 0.$$

This gives $(\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle)^2 \le \|\boldsymbol{x}\|_2^2\|\boldsymbol{y}\|_2^2$, i.e.

$$\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2. \tag{$*$}$$

By definition of $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ it is easy to check that

$$\langle \lambda\boldsymbol{x}, \boldsymbol{y}\rangle = \lambda\langle \boldsymbol{x}, \boldsymbol{y}\rangle \qquad (\lambda \in \mathbf{C}).$$

If $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$ set $\lambda = 1$. Otherwise set $\lambda = \overline{\langle \boldsymbol{x}, \boldsymbol{y}\rangle}/|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|$. In either case $|\lambda| = 1$. With these choices of λ one checks (and please do check) that

$$\langle \lambda\boldsymbol{x}, \boldsymbol{y}\rangle = \lambda\langle \boldsymbol{x}, \boldsymbol{y}\rangle = |\langle \boldsymbol{x}, \boldsymbol{y}\rangle|.$$

From $(*)$ applied to $\lambda\boldsymbol{x}$ and $\boldsymbol{y}$ we get

$$|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| = \mathrm{Re}\langle \lambda\boldsymbol{x}, \boldsymbol{y}\rangle \leq \|\lambda\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 = \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2.$$

The last equality is from the easily checked equality $\|\lambda\boldsymbol{x}\|_2 = |\lambda|\|\boldsymbol{x}\|_2$ (i.e. Property (2)). This establishes (1.1.3.1) and (1.1.3.2). Now in $\mathbf{C}^n$ (and a similar computation works for $\mathbf{R}^n$) we have

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|_2^2 &= \sum_{i=1}^{n}(x_i+y_i)(\bar{x}_i+\bar{y}_i)\\
&= \|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2 + 2\mathrm{Re}\langle \boldsymbol{x}, \boldsymbol{y}\rangle\\
&\leq \|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2 + 2|\langle \boldsymbol{x}, \boldsymbol{y}\rangle| \qquad \text{(By Cauchy-Schwarz)}\\
&\leq \|\boldsymbol{x}\|_2^2 + \|\boldsymbol{y}\|_2^2 + 2\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2\\
&= (\|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2)^2
\end{aligned}$$

This establishes (3) for $\|\ \|_2$.

1.2. **Limits in normed spaces.** Let V and W be normed linear spaces over $\mathbf{K}$. Here are the obvious generalisations of sequences and limits from the real line.

Definition 1.2.1. Let $\{s_n\}$ be a sequence in V. We say $\{s_n\}$ *converges to* $s \in V$ if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$\|s_n - s\| < \epsilon \qquad\qquad (n \geq N)$$

and in this case we write

$$s = \lim_{n\to\infty} s_n$$

and call s the limit of $\{s_n\}$.

Definition 1.2.2. If C is a subset of V, we say $v \in V$ is a *limit point* of C if there is a sequence $\{s_n\}$ in C with $s_n \neq v$ for every $n \in \mathbf{N}$ such that $\lim_{n\to\infty} s_n = v$. Let C' denote the set of limit points of C. The *closure* of C is the set

$$\overline{C} := C \cup C'.$$

(Check that $\overline{C}$ consists of points $v \in V$ such that v is the limit of some some sequence in C.)

Definition 1.2.3. Let C be a subset of V, v a limit point of C, and $f\colon C \to W$ a map (recall W is also a normed space). We say $w \in W$ is the *limit of* $f(x)$ *as* x *approaches* v if the following holds: For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|f(x) - w\| < \epsilon$$

for every $x \in C$ satisfying $0 < \|x - v\| < \delta$. When this happens, we write

$$\lim_{x\to v} f(x) = w.$$

[One could extend this notion to all of $\overline{C}$ (not just limit points) with the caveat that one has to drop the "the" in "the limit". In other words, if $v \in \overline{C}$ is an isolated point (i.e. not a limit point of C), then $\lim_{x\to v} f(x)$ is not unique—in fact it is any number you can think of. This is awkward, and not very useful, which is why the notion is generally restricted to limit points of C.]

Definition 1.2.4. Let $C \subset V$, v a point of C and $f\colon C \to W$ a map. We say f is *continuous at* v if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|f(x) - f(v)\| < \epsilon$$

for every $x \in C$ satisfying $\|x - v\| < \delta$. We say f is continuous on C if it is so at every point of C.

Remark 1.2.5. Let C' denote the set of limit points of C. If $v \in C \cap C'$, then f is continuous at v if and only if $\lim_{x\to v} f(x) = f(v)$. This is clear from the defintions.

Example 1.2.6. Let I be a closed bounded interval and consider the normed space $C(I)$ defined in 1.1.2 (3). Let $\{f_n\}$ be sequence in $C(I)$ converging to f. This means given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$\|f_n - f\|_\infty < \epsilon \qquad\qquad (n \geq N).$$

Now $\|f_n - f\|_\infty = \sup_{y\in I}|f_n(y) - f(y)|$. Hence $\sup_{y\in I}|f_n(y) - f(y)| < \epsilon$ for every $n \geq N$. Now, for every $x \in I$, $|f_n(x) - f(x)| \leq \sup_{y\in I}|f_n(y) - f(y)|$ and hence we have

$$|f_n(x) - f(x)| < \epsilon \qquad\qquad (n \geq N; x \in I).$$

Since N does not depend on x, this means $\{f_n\}$ converges uniformly to f.

Conversely, suppose $\{f_n\}$ is a sequence in $C(I)$ which converges uniformly to a function f on I. From Analysis I you know that f is continuous and hence $f \in C(I)$. Given $\epsilon > 0$ we can find an $N \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \qquad\qquad (n \geq N; x \in I).$$

This gives

$$\sup_{x\in I}|f_n(x) - f(x)| \leq \frac{\epsilon}{2} \qquad\qquad (n \geq N).$$

It follows that

$$\|f_n - f\|_\infty < \epsilon \qquad\qquad (n \geq N).$$

Thus $\{f_n\}$ converges to f in $C(I)$.

We have therefore proven:

Proposition 1.2.7. *Let I be a compact interval in $\mathbf{R}$. A sequence $\{f_n\}$ converges to f in $(C(I), \|\ \|_\infty)$ if and only if it converges uniformly to f on I.*

About these notes. These course notes are a reasonably faithful record of the lectures given at the Chennai Mathematical Institute (CMI) in the January-April semester of 2019-20. The course is Analysis II, a core course for first year undergraduates at CMI. For more material related to this course, visit `https://www.cmi.ac.in/~pramath/teaching.html#ANA2`. If you have comments on these notes, or on related course material, please send an email to pramath@cmi.ac.in.