HW 9

Due on April 13, 2020 (via moodle by 2 pm).

Critical points along a level set. Suppose d, m, n, f, U, $\boldsymbol{c}$, $\boldsymbol{h}$, and M satisfy the hypotheses of Theorem 1.1.1 of Lecture 19. A point $\boldsymbol{a} \in M$ is said to be a *critical point* or a *stationary point* of $f|_M$ if for every $\mathscr{C}^1$ path $\boldsymbol{\gamma}\colon (-\epsilon, \epsilon) \to M$ such that $\boldsymbol{\gamma}^{-1}(\boldsymbol{a}) = \{0\}$, the derivative of $f \circ \boldsymbol{\gamma}$ at 0 vanishes. It is clear that points local extrema of $f|_M$ are critical points and the proof of [Lecture 19, Theorem 1.1.1] shows that $\boldsymbol{a}$ is a critical point of $f|_M$ if and only if $\boldsymbol{\nabla} f(\boldsymbol{a})$ is in the normal space of M at $\boldsymbol{a}$, i.e. if and only if there exist scalars λ_i, $i = 1, \dots, m$ such that $\boldsymbol{\nabla} f(\boldsymbol{a}) = \sum_{i=1}^{m} \lambda_i \boldsymbol{\nabla} h_i(\boldsymbol{a})$.

Use Lagrange multipliers to find the critical points of f subject to the given constraints. The domain of f in each case is the natural domain.

1. $f(x, y, z) = 8x - 4z$; $x^2 + 10y^2 + z^2 = 5$.

2. $f(x, y) = e^{xy}$; $x^3 + y^3 = 16$.

3. $f(x, y, z) = x + y + z$; $x^2 - y^2 = 1$, $2x + z = 1$.

Maxima, minima, and critical points.

4. Let Δ be the distance of $\mathbf{0} \in \mathbf{R}^3$ from the curve C given by the intersection of the cone $x^2 + y^2 = z^2$ with the plane $x - 2z = -5$, i.e. $\Delta = \inf_{\boldsymbol{p} \in C} \|\boldsymbol{p}\|$.
 (a) Find Δ and find the points on C on which are at a distance Δ from $\mathbf{0}$.
 (b) Suppose $\boldsymbol{q}$ is the centre of the ellipse. Set up a problem of constrained maxima and minima to find the lengths of the major and minor axes of the ellipse. You do not have to solve the problem after setting it up.

5. Let Q be a homogeneous degree two polynomial in n variables and M the quadric hypersurface $Q = c$ where $c \neq 0$ is a constant. Assume that M is non-empty and that the Hessian of Q is nonsingular. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$.
 (a) Show that all the eigenvalues of the $n \times n$ symmetric matrix A associated with the quadratic form Q are non zero.
 (b) Show that if $\boldsymbol{a}$ is a stationary point of $f|_M$ then $\boldsymbol{a}$ lies along a principal axis of M, i.e. along a line of support of an eigenvector of the matrix A in part (a).
 (c) Show that every principal axis of M has a stationary point of $f|_M$ on it if and only if all the eigenvalues of the matrix A defined in part (a) have the same sign. (Experiment with standard conic sections and conicoids to get an intuition for the point of this exercise.)

Problems to think about. You don't have submit these.

Let Γ be an orthogonal $n \times n$ matrix, U an open set in $\mathbf{R}^n$ and $V = \Gamma^t(U)$. For a function $f\colon U \to \mathbf{R}$, let $f_\Gamma\colon V \to \mathbf{R}$ be the map $f_\Gamma := f \circ \Gamma$. Similarly, for $\boldsymbol{f} = (f_1, \dots, f_k)\colon U \to \mathbf{R}^k$, $\boldsymbol{f}_\Gamma := \boldsymbol{f} \circ \Gamma$.

6. Let f be $\mathscr{C}^1$. Show that $\boldsymbol{a} \in V$ is a critical point of f_Γ if and only if $\Gamma\boldsymbol{a}$ is a critical point of f.

7. Let $m \leq n$ be a non-negative integer, $\boldsymbol{h}\colon U \to \mathbf{R}^m$ a $\mathscr{C}^1$ map, $\boldsymbol{c}$ a point in $\boldsymbol{h}(U)$, M the level set $\boldsymbol{h}^{-1}(\boldsymbol{c})$ satisfying the condition that $\operatorname{rk} \boldsymbol{h}'(\boldsymbol{x}) = m$ for all $\boldsymbol{x} \in M$. Let $M_\Gamma = \Gamma^t(M)$.
   (a) Show that $M_\Gamma = \boldsymbol{h}_\Gamma^{-1}(\boldsymbol{c})$ and that $\operatorname{rk} \boldsymbol{h}_\Gamma'(\boldsymbol{x}) = m$ for all $\boldsymbol{x} \in M_\Gamma$.
   (b) Show that $\boldsymbol{a} \in M_\Gamma$ is a critical point of $f_\Gamma|_{M_\Gamma}$ if and only if $\Gamma\boldsymbol{a}$ is a critical point of $f|_M$.

8. Find the maxima and minima of $f(x, y, z) = \frac{1}{\sqrt{2}}(x - y)$ subject to the constraints

$$3x^2 + 3y^2 + 2z^2 + 2\sqrt{2}zx + 2\sqrt{2}yz - 2xy = 16 \quad \text{and} \quad 2yz + 2zx - x + y = 0.$$

Relate it to a problem solved in the lecture notes. Use Problem 7.

9. Consider the ellipse C described in Problem 4. Show that its centre is $\boldsymbol{q} = (5/3, 0, 10/3)$.

Here are some graphics for Problem 9:

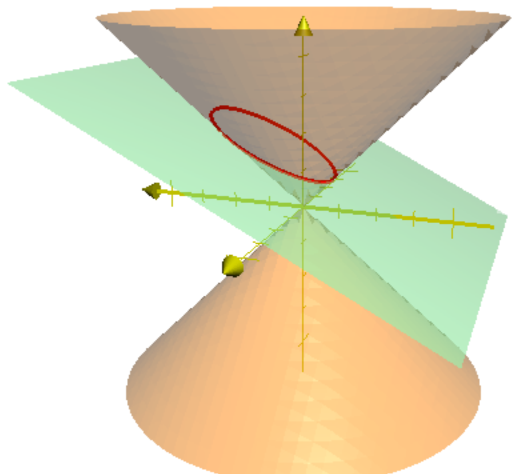

FIGURE 1. The green plane is $x - 2y = -5$ and the cone is $x^2 + y^2 = z^2$. The red curve is their intersection C. It is an ellipse. Identify the x-axis.

The next two pages contain one picture each.

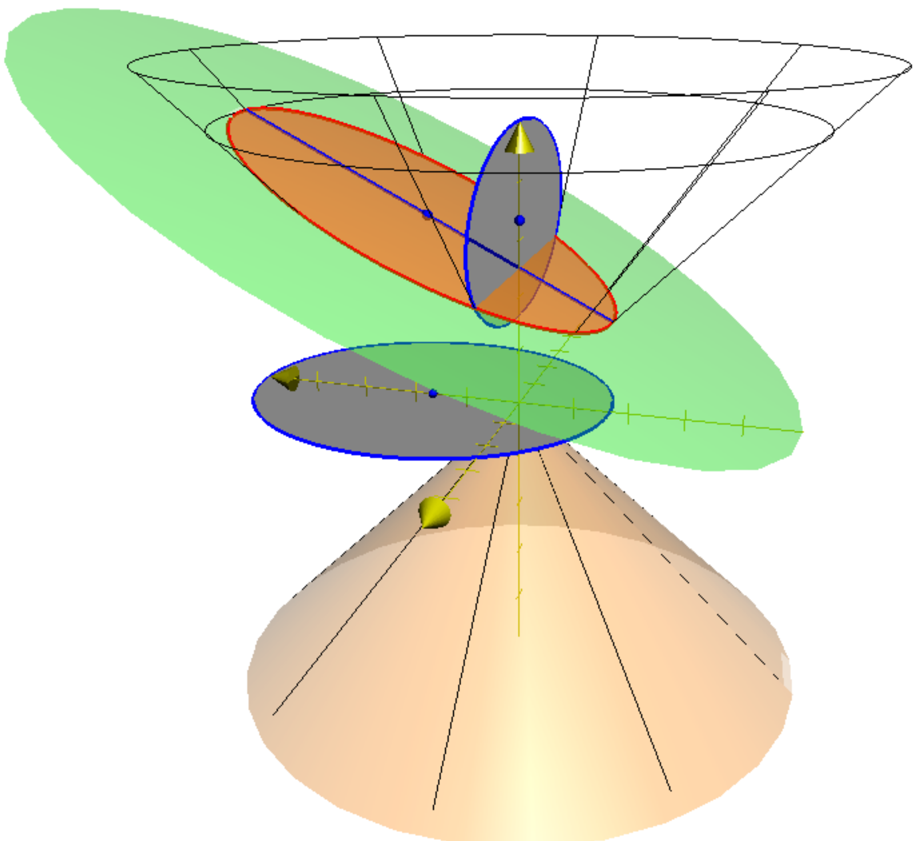

FIGURE 2. In this picture the elliptical region enclosed by C is shaded in a tan colour. Its projections to the xy-plane and the yz-plane are shaded grey. Their boundaries (in blue) are also ellipses (and by eliminating a variable, you should be able to write down their equations in the appropriate plane). The centres of all three ellipses are also marked. The next page contains another angle.

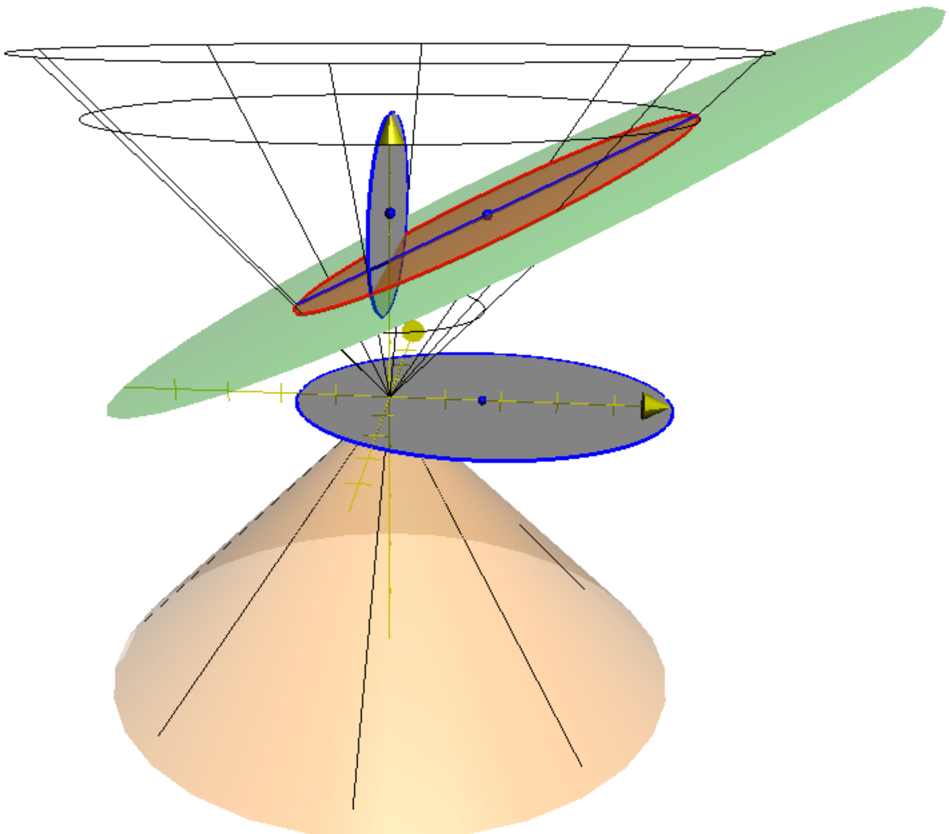

FIGURE 3. This is another angle of the last picture. Where is the x-axis?