# HW 8

Due on April 6, 2020 (via moodle by 2 pm).

**Conicoids.** This is the classical name for quadric surfaces. We will be confining ourselves to conicoids in $\mathbf{R}^3$, i.e. surfaces $S$ of the form $F(x,y,z)=0$ where $F$ is a polynomial of degree two in three variables over $\mathbf{R}$. A *standard ellipsoid* over $\mathbf{R}$ is one which has an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

A *standard hyperboloid of one sheet* is one which has an equation of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

A *standard hyperboloid of two sheets* is one which has an equation of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

An *ellipsoid (respectively a hyperboloid of one sheet, respectively a hyperboloid of two sheets)* is a conicoid which after a translation (i.e. a change of origin) and an orthogonal transformation is a standard ellipsoid (respectively hyperboloid of one sheet, respectively a hyperboloid of two sheets). There are figures in the last two pages to help you.

**Taylor's series.**

**1.** Suppose $f$ is a polynomial in $n$-variables over $\mathbf{R}$ of degree at most $m$ (the degree of the zero polynomial will be regarded as $-\infty$). Suppose

$$\lim_{\boldsymbol{x}\to\mathbf{0}} |f(\boldsymbol{x})|/\|\boldsymbol{x}\|^m = 0.$$

Show that $f(\boldsymbol{x})=0$ for all $\boldsymbol{x}\in\mathbf{R}^n$.

**2.** Let $f$ be a degree two polynomial over $\mathbf{R}$ in $n$-variables and $\boldsymbol{a}=(a_1,\dots,a_n)$ a critical point of $f$. Show that there is a homogenous degree two polynomial $Q$ such that

$$f(\boldsymbol{x})-f(\boldsymbol{a})=Q(\boldsymbol{x}-\boldsymbol{a}) \qquad (\boldsymbol{x}\in\mathbf{R}^n).$$

**Principal axes and principal directions of a conicoid.** Here is some (very very) classical algebraic geometry.[1]

In Example 2.2.5 of Lecture 15 we showed that the curve $C$ in $\mathbf{R}^2$ with equation $x^2-yx+y^2=3$ is an ellipse. Please look at the example again. The lines $\ell_1=\{(t,t)\mid t\in\mathbf{R}\}$ and $\ell_2=\{t(-1,1)\mid t\in\mathbf{R}\}$ are called the *principal axes of $C$* and the ratios $1:1$ and $-1:1$ are classically called the *prinicipal directions*. Today they are identified (up to sign) with the unit vectors in the directions of $\ell_1$ and $\ell_2$.

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[1] See the classic: *An Elementary Treatise on Coordinate Geometry of Three Dimensions*, by R.J.T. Bell (published in 1910). It is the book from which generations learned the elementary parts of the subject. Have a crack at the book (you don't have to submit that exercise).

In $\mathbf{R}^3$, let $S$ be a conicoid, say given by $F(x, y, z) = c$, where $F$ is a degree two polynomial (sometimes the "surface" is empty as in $x^2 + y^2 + z^2 = -1$ or has only one point as in $x^2 + y^2 + z^2 = 0$, but these are exceptional and for ease of discussion we will regard the locus of $F(x, y, z) = c$ as a surface). If the equation of $S$ can be re-written as

$$\begin{aligned} A(x-a)^2 + B(y-b)^2 + C(z-c)^2 + D(y-b)(z-c) \\ + E(z-c)(x-a) + F(x-a)(y-b) = \varrho, \end{aligned}$$

then $(a, b, c)$ is called a *centre* of the conicoid $S$. We are not changing coordinates when re-expressing $S$ in this way. Note that if we move the origin to the centre of the conicoid, the equation becomes simpler.

If $S$ can be written as

$$\sum_{j=1}^{3} \lambda_i (\gamma_{1j} x + \gamma_{2j} y + \gamma_{3j} z)^2 + \sum_{j=1}^{3} \mu_i (\gamma_{1j} x + \gamma_{2j} y + \gamma_{3j} z) = c,$$

with the vectors $(\gamma_{1_j}, \gamma_{2_j}, \gamma_{3_j})$, $j = 1, 2, 3$, forming an orthonormal bases, then these three vectors (up to sign) are called a set of *principal directions* of $S$ and the lines $\ell_j = \{(a, b, c) + t(\gamma_{1_j}, \gamma_{2_j}, \gamma_{3_j}) \mid t \in\}$, $j = 1, 2, 3$, a set of *principal axes* of $S$, where $(a, b, c)$ is a centre of $S$. A *principal axis* is a member of a set of principal axes.

**3**. Let $f$ be a polynomial function of degree two in three variables over $\mathbf{R}$, and suppose the Hessian $H$ is non-singular. Assume the locus $S$ of points satusfying $f(x, y, z) = c$ has more than one point. Show that $S$ has a unique centre.

**4**. Find a centre and a set of principal axes of the following conicoids. Are they ellipsoids, hyperboloids (of how many sheets?), or something else?
(a) $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$
(b) $3x^2 + 7y^2 + 3z^2 + 10yz - 2zx + 10xy + 4x - 12y + 4z + 1 = 0$.
(c) $11y^2 + 14yz + 8zx + 14xy - 6x - 16y + 2z - 2 = 0$

**Critical points.**

**5**. Find the critical points of the following functions, and classify them as degenerate or non-degenerate. If the latter, further classify them as local maxima, local minima, or as saddle points.
(a) $f(x, y) = x^4 + y^4 - 4xy + 1$.
(b) $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.
(c) $f(x, y) = e^{4y - x^2 - y^2}$.

**6**. Find the critical points of the functions in Problem 4. Classify them as degenerate or non-degenerate. If the latter, further classify them as local maxima, local minima, or as saddle points.

The next two pages have figures of an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets.

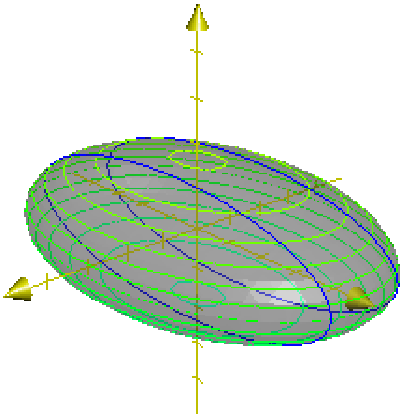

FIGURE 1. Ellipsoid

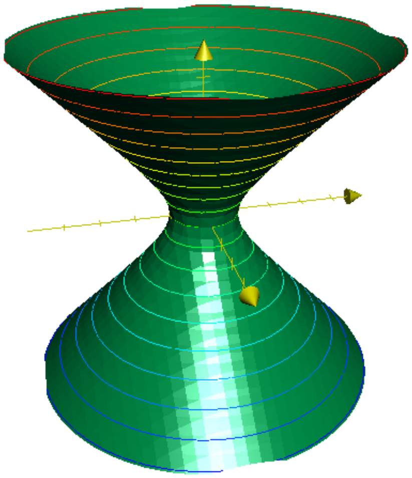

FIGURE 2. Hyperboloid of one sheet.

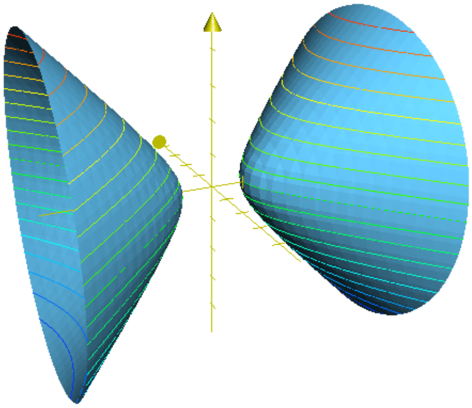

Figure 3. Hyperboloid of two sheets.